Strictly Competitive Games

- A strictly competitive or zero-sum game is a 2-player strategic game such that for each $a \in A$, we have $u_1(a) + u_2(a) = 0$.
  - What is good for me, is bad for my opponent and vice versa.
- Note: Any game where the sum is a constant $c$ can be transformed into a zero-sum game with the same set of equilibria:
  - $u_i'(a) = u_i(a)$
  - $u_j'(a) = u_j(a) - c$

How to Play Zero-Sum Games?

- Assume that only pure strategies are allowed
  - Dominating strategy?
  - Nash equilibrium?
- Be paranoid: Try to minimize your loss by assuming the worst!
- Player 1 takes minimum over row values:
  - T: -6, M: -1, B: -6
  - then maximizes:
    - M: -1

Maximinimizer

- An action $x^*$ is called maximinimizer for player 1, if
  $$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y)$$
  for all $x \in A_1$.
- Similar for player 2
- Maximinimizer try to minimize the loss, but do not necessarily lead to a Nash equilibrium.
- However, if a NE exists, then the action profile is a pair of maximinimizers!

Maximinimizer Theorem

In strictly competitive games:
1. If $(x^*, y^*)$ is a Nash equilibrium of $G$ then $x^*$ is a maximinimizer for player 1 and $y^*$ is a maximinimizer for player 2.
2. If $(x^*, y^*)$ is a Nash equilibrium of $G$ then $\max_y \min_x u_i(x, y) = \min_y \max_x u_i(x, y) = u_i(x^*, y^*)$.
3. If $\max_x \min_y u_i(x, y) = \min_x \max_y u_i(x, y)$ and $x^*$ is a maximinimizer for player 1 and $y^*$ is a maximinimizer for player 2, then $(x^*, y^*)$ is a Nash equilibrium.

Some Consequences

- Because of (2): if $(x^*, y^*)$ is a NE then $\max_x \min_y u_i(x, y) = u_i(x^*, y^*)$, all NE yield the same payoff.
  - It is irrelevant which we choose.
- Because of (2), if $(x^*, y^*)$ and $(x', y')$ are a NEs then $x^*$, $x'$ are maximinimizers for player 1 and $y^*$, $y'$ are maximinimizers for player 2. Because of (3), then $(x^*, y^*)$ and $(x', y')$ are NEs as well!
  - It is not necessary to coordinate in order to play in a NE!
Example

- Minimum in rows (for player 1):
  - T: -6, M: -1, B: -6
- Maximinimizer:
  - M: -1
- Maximum over columns (for player 1)
  - L: 8, M: -1, R: 8
- Minimaximizer:
  - M: -1
- Also NE, apparently

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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<td>-3, 3</td>
<td>-6, 6</td>
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<tr>
<td>M</td>
<td>2, -2</td>
<td>-1, 1</td>
<td>3, -3</td>
</tr>
<tr>
<td>B</td>
<td>-6, 6</td>
<td>-4, 4</td>
<td>8, -8</td>
</tr>
</tbody>
</table>

How to Find NEs in Mixed Strategies?

- While it is non-trivial to find NEs for general sum games, zero-sum games are "easy"
- Let's test all mixed strategies of player 1 $\alpha_1$ against all mixed strategies of player 2 $\alpha_2$. Then use only those that are maximinimizers.
- Since all mixed strategies are linear combinations of pure strategies, it is enough to check against the pure strategies of player 2 (support theorem).
- We just have to optimize, i.e., find the best mixed strategy ➢ Use linear programming

Linear Programming: The Idea

- The article-mix problem:
  - article 1 needs: 25 min of cutting, 60 min of assembly, 68 min of postprocessing
    - results in 30 Euro profit per article
  - article 2 needs: 75 min of cutting, 60 min of assembly, and 34 min of postprocessing
    - results in 40 Euro profit per article
  - per day: 450 min of cutting, 480 min of assembly and 476 min of postprocessing
- Try to maximize profit

Linear Programming: The General Case

- $n$ real-valued variables $x_i$
- $m$ coefficients $b_i$ and constants $c_j$
- $m \cdot n$ coefficients $a_{ij}$
- $m$ equations $\sum_i a_{ij} x_i = c_j$
- objective function: $\sum_i b_i x_i$ is to be minimized
- Can be solved by the simplex method ➢ lpsolve (under Linux)

Feasible Solutions

- The inequalities describe convex sets in $\mathbb{R}^2$
- The intersection of all convex sets represents the set of feasible solutions
- Each point in the set of feasible solutions could get a quality measure according to the objective function
- Consider lines of equal quality and then do hill climbing!

Resulting Constraints & Optimization Goals

- $x$: #article1, $y$: #article2
- $x \geq 0, y \geq 0$
- $25x + 75x \leq 450$ (cutting)
  ➢ $y \leq 6 - \frac{1}{3}x$
- $60x + 60y \leq 480$ (assembly)
  ➢ $y \leq 8 - x$
- $68x + 34y \leq 476$ (postprocessing)
  ➢ $y \leq 14 - 2x$
- Maximize $z = 30x + 40y$
Solving Zero-Sum Games

• Let $A_1 = \{a_{11}, \ldots, a_{1n}\}, A_2 = \{a_{21}, \ldots, a_{2m}\}$.
• Player 1 looks for a mixed strategy $\alpha_1$
  $- \sum_j \alpha_1(a_{1j}) = 1$
  $-\alpha_1(a_{1j}) \geq 0$
  $- \sum_j \alpha_1(a_{1j}) \cdot u_1(a_{1j}, a_{2i}) \geq u$ for all $i \in \{1, \ldots, m\}$
  $-\text{Maximize } u$
• Similarly for player 2.

Conclusion

• Zero-sum games are particularly simple
• Playing a pure maximinizing strategy minimizes loss (for pure strategies)
• If NE exists, it is a pair of maximinimizers
• NEs can be freely “mixed”
• In mixed strategies, NEs always exists
• Can be determined by linear programming