An Introduction to Game Theory
Part II: Mixed and Correlated Strategies
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Randomizing Actions ...
- Since there does not seem to exist a rationale decision, it might be best to randomize strategies.
- Play Head with probability $p$ and Tail with probability $1-p$
- Switch to expected utilities

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<tr>
<td>Head</td>
<td>1,-1</td>
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<tr>
<td>Tail</td>
<td>-1,1</td>
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Some Notation
- Let $G = (N, (A_i), (u_i))$ be a strategic game
- Then $\Delta(A_i)$ shall be the set of probability distributions over $A_i$ – the set of mixed strategies $\alpha_i \in \Delta(A_i)$
- $\alpha_i(a_i)$ is the probability that $a_i$ will be chosen in the mixed strategy $\alpha_i$
- A profile $\alpha = (\alpha_i)$ of mixed strategies induces a probability distribution on $A$: $p(a) = \prod_i \alpha_i(a_i)$
- The expected utility is $U_i(\alpha) = \sum_{a \in A} p(a) \cdot u_i(a)$

Example of a Mixed Strategy
- Let
  - $\alpha_1(H) = 2/3$, $\alpha_1(T) = 1/3$
  - $\alpha_2(H) = 1/3$, $\alpha_2(T) = 2/3$
- Then
  - $p(H,H) = 2/9$
  - ...
  - $U_1(\alpha_1, \alpha_2) = ?$
  - $U_2(\alpha_1, \alpha_2) = ?$

Mixed Extensions
- The mixed extension of the strategic game $(N, (A_i), (u_i))$ is the strategic game $(N, \Delta(A_i), (U_i))$.
- The mixed strategy Nash equilibrium of a strategic game is a Nash equilibrium of its mixed extension.
- Note that the Nash equilibria in pure strategies (as studied in the last part) are just a special case of mixed strategy equilibria.

Nash’s Theorem
Theorem. Every finite strategic game has a mixed strategy Nash equilibrium.
- Note that it is essential that the game is finite
- So, there exists always a solution
- What is the computational complexity?
- This is an open problem! Not known to be NP-hard, but there is no known polynomial time algorithm
The Support

- We call all pure actions $a_i$ that are chosen with non-zero probability by $a_i$ the support of the mixed strategy $a_i$

**Lemma.** Given a finite strategic game, $a_i^*$ is a mixed strategy equilibrium if and only if for every player $i$ every pure strategy in the support of $a_i^*$ is a best response to $a_i^*$.

Proving the Support Lemma

Assume that $a_i^*$ is a Nash equilibrium with $a_i$ being in its support but not being a best response to $a_i^*$.

- This means, by reassigning the probability of $a_i$ to the other actions in the support, one can get a higher payoff for player $i$.
- This implies $a_i^*$ is not a Nash equilibrium $\Rightarrow$ (Proving the contraposition): Assume that $a_i^*$ is not a Nash equilibrium.
- This means that there exists $a_i’$ that is a better response than $a_i^*$ to $a_i^*$.
- Then because of how $U_i$ is computed, there must be an action $a_i’$ in the support of $a_i’$ that is a better response (higher utility) to $a_i^*$ than an action $a_i’$ in the support of $a_i^*$.
- This implies that there are actions in the support of $a_i^*$ that are not best responses to $a_i^*$.

Using the Support Lemma

- The Support Lemma can be used to characterize all Nash equilibria types in 2-person 2x2 action games.
- There are 4 potential Nash equilibria in pure strategies
  - Easy to check
  - There are another 4 potential Nash equilibria types with a 1-support (pure) against 2-support mixed strategies
    - Exists only if the payoff for the mixed strategy player is identical for both pure strategies and one of the corresponding pure strategy profiles is already a Nash equilibrium (follows from Support Lemma)
  - There exists one other potential Nash equilibrium type with a 2-support against a 2-support mixed strategy
    - Here we can use the Support Lemma to compute the NE (if there exists one)

A Mixed Nash Equilibrium for Matching Pennies

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- There is clearly no NE in pure strategies
- Let's try whether there is a 2/2 NE $\alpha^*$ in mixed strategies
- Then the H action by player 1 should have the same utility as the T action when played against the mixed strategy $\alpha^*$

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| $U((1,0), (a_2(T), a_2(T)))$ | $U((0,1), (a_2(H), a_2(T)))$ | $U((0,0), (a_2(H), a_2(T)))$
| $U((1,0), (a_2(H), a_2(T)))$ | $U((0,0), (a_2(H), a_2(T)))$ | $U((0,1), (a_2(H), a_2(T)))$
| $U((0,0), (a_2(H), a_2(T)))$ | $U((0,1), (a_2(H), a_2(T)))$ | $U((1,0), (a_2(H), a_2(T)))$
| $a_2(H)$ = $a_2(T)$ | $a_2(H)$ | $a_2(T)$
| $a_2(H)$ | $a_2(T)$ | $a_2(H)$
| $a_2(H)$ | $a_2(T)$ | $a_2(H)$
| $a_2(T)$ | $a_2(H)$ | $a_2(T)$

- Because of $a_2(H)$ or $a_2(T)$ = 1:
  - $a_2(H)$ or $a_2(T)$ = 1/2
  - Similarly for player $i$!
- $U_i(\alpha^*) = 0$

Mixed NE for BoS

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<tr>
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<td>0,0</td>
<td>1,2</td>
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- There are obviously 2 NEs in pure strategies
- Is there also a 2/2 strictly mixed NE?
- If so, again B and S played by player 1 should lead to the same payoff.

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| $U_1((1,0), (a_2(B), a_2(S)))$ = $U_1((1,0), (a_2(B), a_2(S)))$ | $U_1((0,1), (a_2(B), a_2(S)))$ | $U_1((0,0), (a_2(B), a_2(S)))$
| $U_1((1,0), (a_2(B), a_2(S)))$ | $U_1((0,1), (a_2(B), a_2(S)))$ | $U_1((0,0), (a_2(B), a_2(S)))$
| $a_2(B)$ = $a_2(S)$ | $a_2(B)$ | $a_2(S)$
| Because of $a_2(B)$ or $a_2(S)$ = 1:
  - $a_2(B)$ or $a_2(S)$ = 1/2
  - Similarly for player $i$!
- $U_i(\alpha^*) = 2/3$

Couldn’t we Help the BoS Players?

- BoS have two pure strategy Nash equilibria
  - but which should they play?
- They can play a mixed strategy, but this is worse than any pure strategy
- The solution is to talk about, where to go
- Use an external random signal to decide where to go
- Correlated Nash equilibria
- In the BoS case, we get a payoff of 1.5
Conclusion

- Although Nash equilibria do not always exist, one can give a guarantee, when we randomize finite games:
  - For every finite strategic game, there exists a Nash equilibrium in mixed strategies.
- Actions in the support of mixed strategies in a NE are always best answers to the NE profile, and therefore have the same payoff → Support Lemma.
- The Support Lemma can be used to determine mixed strategy NEs for 2-person games with 2x2 action sets.
- In general, there is no poly-time algorithm known for computing a Nash equilibrium (and it is open whether this problem is NP-hard).
- In addition to pure and mixed NEs, there exists the notion of correlated NE, where you coordinate your action using an external randomized signal.