### Strategic Game

- A strategic game $G$ consists of
  - a finite set $N$ (the set of players)
  - for each player $i \in N$ a non-empty set $A_i$ (the set of actions or strategies available to player $i$), whereby $A = \bigcup_{i} A_i$
  - for each player $i \in N$ a function $u_i : A \to \mathbb{R}$ (the utility or payoff function)
  - $G = (N, (A_i), (u_i))$
- If $A$ is finite, then we say that the game is **finite**

### Playing the Game

- Each player $i$ makes a decision which action to play: $a_i$
- All players make their moves simultaneously leading to the action profile $a^* = (a_1, a_2, \ldots, a_n)$
- Then each player gets the payoff $u_i(a^*)$
- Of course, each player tries to maximize its own payoff, but what is the right decision?
  - **Note:** While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
  - If we want to model something like this, the payoff function must be changed

### Notation

- For 2-player games, we use a matrix, where the strategies of player 1 are the rows and the strategies of player 2 the columns
- The payoff for every action profile is specified as a pair $x, y$, whereby $x$ is the value for player 1 and $y$ is the value for player 2
- Example: For $(T, R)$, player 1 gets $x_{12}$, and player 2 gets $y_{12}$

### Example Game: Bach and Stravinsky

- Two people want to go together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers, the other Stravinsky. **Will they meet?**
- This game is also called the **Battle of the Sexes**

<table>
<thead>
<tr>
<th></th>
<th>Bach</th>
<th>Stravinsky</th>
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<tbody>
<tr>
<td><strong>Bach</strong></td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td><strong>Stravinsky</strong></td>
<td>0,0</td>
<td>1,2</td>
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### Example Game: Hawk-Dove

- Two animals fighting over some prey.
- Each can behave like a dove or a hawk
- The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove
- This game is also called **chicken**.

<table>
<thead>
<tr>
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<tr>
<td><strong>Dove</strong></td>
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Example Game: Prisoner’s Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

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Solving a Game

- What is the right move?
- Different possible solution concepts
  - Elimination of strictly or weakly dominated strategies
  - Maximin strategies (for minimizing the loss in zero-sum games)
  - Nash equilibrium
- How difficult is it to compute a solution?
- Are there always solutions?
- Are the solutions unique?

Strictly Dominated Strategies

- Notation:
  - Let \( a = (a_i) \) be a strategy profile
  - \( a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n) \)
  - \( (a_{-i}, a'_i) = (a_1, ..., a_{i-1}, a'_i, a_{i+1}, ..., a_n) \)
- Strictly dominated strategy:
  - An strategy \( a_j^* \in A_j \) is strictly dominated if there exists a strategy \( a_j' \) such that for all strategy profiles \( a \in A: u_j(a_{-j}, a_j') > u_j(a_{-j}, a_j^*) \)
- Of course, it is not rational to play strictly dominated strategies

Iterated Elimination of Strictly Dominated Strategies

- Since strictly dominated strategies will never be played, one can eliminate them from the game
- This can be done iteratively
- If this converges to a single strategy profile, the result is unique
- This can be regarded as the result of the game, because it is the only rational outcome

Iterated Elimination: Example

- Eliminate:
  - b4, dominated by b3
  - a4, dominated by a1
  - b3, dominated by b2
  - a1, dominated by a2
  - b1, dominated by b2
  - a3, dominated by a2
- Result: (a2,b2)

Iterated Elimination: Prisoner’s Dilemma

- Player 1 reasons that “not confessing” is strictly dominated and eliminates this option
- Player 2 reasons that player 1 will not consider “not confessing”. So he will eliminate this option for himself as well
- So, they both confess
Weakly Dominated Strategies

- Instead of strict domination, we can also go for weak domination:
  - An strategy \( a_j^* \in A_j \) is **weakly dominated** if there exists a strategy \( a_j \) such that for all strategy profiles \( a \in A \):
    \[
    u_j(a_j^*, a) \geq u_j(a_j, a) \\
    u_j(a_j^*, a) > u_j(a_j, a^*).
    \]

Results of Iterative Elimination of Weakly Dominated Strategies

- The result is not necessarily unique
- Example:
  - Eliminate
    - T (SM)
    - L (SR)
    - Result: (1,1)
  - Eliminate
    - B (SM)
    - R (SL)
    - Result: (2,1)

Existence of Dominated Strategies

- Dominating strategies are a convincing solution concept
- Unfortunately, often dominated strategies do not exist
- What do we do in this case?  
  - Nash equilibrium

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Nash Equilibrium

- A **Nash equilibrium** is an action profile \( a^* \in A \) with the property that for all players \( i \in N \):
  \[
  u_i(a^*) = u_i(a^*, a) \forall a_j \in A_j
  \]
- In words, it is an action profile such that there is no incentive for any agent to deviate from it
- While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept
- If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a Nash equilibrium

Example Nash-Equilibrium: Prisoner's Dilemma

- Don’t – Don’t  - not a NE
- Don’t – Confess (and vice versa)  - not a NE
- Confess – Confess  - NE

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Example Nash-Equilibrium: Hawk-Dove

- Dove-Dove:  - not a NE
- Hawk-Hawk  - not a NE
- Dove-Hawk  - is a NE
- Hawk-Dove  - is, of course, another NE
- So, NEs are not necessarily unique

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Auctions

- An object is to be assigned to a player in the set \{1, \ldots, n\} in exchange for a payment.
- Players' evaluation of the object is \(v_i\), and \(v_1 > v_2 > \ldots > v_n\).
- The mechanism to assign the object is a sealed-bid auction: the players simultaneously submit bids (non-negative real numbers).
- The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment.
- The payment for a first price auction is the highest bid.
- What are the Nash equilibria in this case?

Formalization

- Game \(G = (\{1, \ldots, n\}, (A_i), (u_i))\)
- \(A_i\): bids \(b_i \in \mathbb{R}^+\)
- \(u(b_i, b) = v_i - b_i\) if \(i\) has won the auction, 0 otherwise
- Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.

Nash Equilibria for First-Price Sealed-Bid Auctions

- The Nash equilibria of this game are all profiles \(b\) with:
  - \(b_i \leq b_i\) for all \(i \in \{2, \ldots, n\}\)
    - No \(i\) would bid more than \(v_i\) because it could lead to negative utility.
    - If a bid \(< v_i\) is higher than \(b_i\), player 1 could increase its utility by bidding \(v_i + \epsilon\).
  - \(v_i \geq b_i \geq v_j\)
    - Otherwise, player 1 either loses the bid (and could increase its utility by bidding more) or would have itself negative utility.
  - \(b_i = b_1\) for at least one \(j \in \{2, \ldots, n\}\)
    - Otherwise player 1 could have gotten the object for a lower bid.

Another Game: Matching Pennies

- Each of two people chooses either Head or Tail. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.
- This is also a zero-sum or strictly competitive game.
- No NE at all! What shall we do here?

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<tr>
<th></th>
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<th>Tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
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Conclusions

- Strategic games are one-shot games, where everybody plays its move simultaneously.
- The game outcome is the action profile resulting from the individual choices.
- Each player gets a payoff based on its payoff function and the resulting action profile.
- Iterated elimination of strictly dominated strategies is a convincing solution concept, but unfortunately, most of the time it does not yield a unique solution.
- Nash equilibrium is another solution concept: Action profiles, where no player has an incentive to deviate.
- It also might not be unique and there can be even infinitely many NEs.
- Also, there is no guarantee for the existence of a NE.