An Introduction to Game Theory Part I: Strategic Games

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### Strategic Game

- A strategic game G consists of
  - a finite set N (the set of players)
  - for each player  $i \in N$  a non-empty set  $A_i$  (the set of actions or strategies available to player *i*), whereby  $A = \prod_i A_i$
  - for each player  $i \in N$  a function  $u_i: A \to \mathbb{R}$  (the utility or payoff function)
  - $-G = (N, (A_i), (U_i))$

• If A is finite, then we say that the game is *finite* 

### Playing the Game

- Each player *i* makes a decision which action to play: a,
- All players make their moves simultaneously leading to the action profile  $a^* = (a_1, a_2, ..., a_n)$
- Then each player gets the payoff  $u(a^*)$
- Of course, each player tries to maximize its own payoff, but what is the right decision?
- Note: While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
  - If we want to model something like this, the payoff function must be changed

Notation			
For 2-player games, we use a matrix, where the strategies of player 1 are the rows and the strategies of player 2 the		Player 2 L action	Player 2 <b>R</b> action
columns The payoff for every action profile is specified as a pair $x, y$ , whereby $x$ is the value for player 1 and y is the value for player 2 Example: For (T,R), player 1 gets $x_{12}$ , and player 2 gets $y_{12}$	Player1 <b>T</b> action	x <sub>11</sub> ,y <sub>11</sub>	x <sub>12</sub> ,y <sub>12</sub>
	Player1 <b>B</b> action	x <sub>21</sub> ,y <sub>21</sub>	x <sub>22</sub> ,y <sub>22</sub>

Example Game: Bach and Stravinsky				
<ul> <li>Two people want to out together to a concert of music by either Bach or Stravinsky. Their main</li> </ul>		Bach	Stra- vinsky	
concern is to go out together, but one prefers, the other Stravinsky. Will they meet?	Bach	2,1	0,0	
• This game is also called the <i>Battle of the Sexes</i>	Stra- vinsky	0,0	1,2	

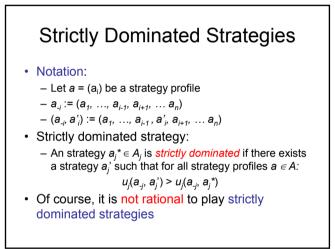
	Dove	Hawk
Dove	3,3	1,4
Hawk	4,1	0,0
		3,3 Hawk

### Example Game: Prisoner's Dilemma

<ul> <li>Two suspects in a crime are put into separate cells.</li> <li>If they both confess, each</li> </ul>		Don't confess	Confess
<ul><li>will be sentenced to 3 years in prison.</li><li>If only one confesses, he will be freed.</li></ul>	Don't confess	3,3	0,4
<ul> <li>If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.</li> </ul>	Confess	4,0	1,1

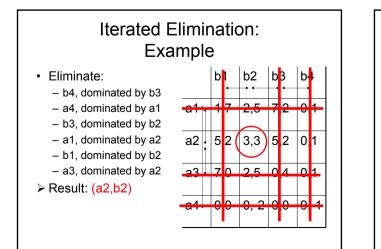
# Solving a Game

- What is the right move?
- Different possible solution concepts
  - Elimination of strictly or weakly dominated strategies
  - Maximin strategies (for minimizing the loss in zerosum games)
  - Nash equilibrium
- How difficult is it to compute a solution?
- · Are there always solutions?
- · Are the solutions unique?



### Iterated Elimination of Strictly Dominated Strategies

- Since strictly dominated strategies will never be played, one can eliminate them from the game
- · This can be done iteratively
- If this converges to a single strategy profile, the result is unique
- This can be regarded as the result of the game, because it is the only rational outcome



#### Iterated Elimination: Prisoner's Dilemma Player 1 reasons that "not Dor 't Confess confessing" is strictly confess dominated and eliminates this option Don't Player 2 reasons that player 1 will not consider confess 04"not confessing". So he will eliminate this option for himself as well Confess · So, they both confess 4.0 1,1

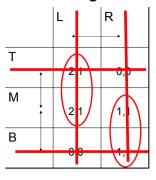
# Weakly Dominated Strategies

- Instead of strict domination, we can also go for weak domination:
  - An strategy  $a_j^* \in A_j$  is *weakly dominated* if there exists a strategy  $a_j^*$  such that for all strategy profiles  $a \in A$ :

$$\begin{split} u_j(a_{-j}, a_j') &\geq u_j(a_{-j}, a_j^*) \\ \text{and for at least one profile } a \in A: \\ u_j(a_{-j}, a_j') &> u_j(a_{-j}, a_j^*). \end{split}$$

### Results of Iterative Elimination of Weakly Dominated Strategies

- The result is not necessarily unique
- Example:
  - Eliminate
    - T (≤M)
    - L (≤R) ≻Result: (1,1)
  - Eliminate:
  - B (≤M)
  - R (≤L)
    - ≻ Result (2,1)



Existence of Dominated Strategies				
<ul> <li>Dominating strategies are a convincing solution concept</li> </ul>		Dove	Hawk	
<ul> <li>Unfortunately, often dominated strategies do not exist</li> </ul>	Dove	3,3	1,4	
<ul> <li>What do we do in this case?</li> <li>▶ Nash equilibrium</li> </ul>	Hawk	4,1	0,0	
		I		

# Nash Equilibrium A Nash equilibrium is an action profile a\* ∈ A with the property that for all players i ∈ N: u<sub>i</sub>(a\*) = u<sub>i</sub>(a\*<sub>-i</sub>, a\*<sub>i</sub>) ≥ u<sub>i</sub>(a\*<sub>-i</sub>, a<sub>i</sub>) ∀ a<sub>i</sub> ∈ A<sub>i</sub> In words, it is an action profile such that there is no incentive for any agent to deviate from it While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a Nash equilibrium

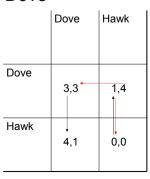
Example Nash-Equilibrium: Prisoner's Dilemma			
<ul> <li>Don't – Don't</li> <li>not a NE</li> <li>Don't – Confess (and</li> </ul>		Don't confess	Confess
vice versa) – not a NE • Confess – Confess	Don't confess	3,3	0,4
– NE	Confess	4,0	1,1

## Example Nash-Equilibrium: Hawk-Dove

- Dove-Dove:
   not a NE
- Hawk-Hawk

   not a NE
- Dove-Hawk
   is a NE
- Hawk-Dove

   is, of course, another NE
- So, NEs are not necessarily unique



### **Auctions**

- An object is to be assigned to a player in the set {1,...,n} in exchange for a payment.
- Players *i* evaluation of the object is  $v_i$  and  $v_1 > v_2 > ... >$ V<sub>n</sub>.
- The mechanism to assign the object is a sealed-bid auction: the players simultaneously submit bids (nonnegative real numbers)
- The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment
- The payment for a *first price* auction is the highest bid.
- What are the Nash equilibria in this case?

### Formalization

- Game G =  $(\{1, ..., n\}, (A_i), (u_i))$
- $A_i$ : bids  $b_i \in \mathbb{R}^+$
- $u_i(b_{-i}, b_i) = v_i b_i$  if *i* has won the auction, 0 othwerwise
- Nobody would bid more than his valuation, because this could lead to negative utility. and we could easily achieve 0 by bidding 0

### Nash Equilibria for First-Price Sealed-Bid Auctions

- The Nash equilibria of this game are all profiles b with:
  - $-b_i \le b_1$  for all  $i \in \{2, ..., n\}$ 
    - No *i* would bid more than  $v_2$  because it could lead to negative utility
    - If a  $\dot{b}_1$  (with <  $v_2$ ) is higher than  $b_1$  player 1 could increase its utility by bidding  $v_2 + \varepsilon$

    - So i wins in all NEs
  - $-V_1 \ge b_1 \ge V_2$ 
    - Otherwise, player 1 either looses the bid (and could increase its utility by bidding more) or would have itself negative utility
  - $-b_i = b_1$  for at least one  $j \in \{2, ..., n\}$ 
    - Otherwise player 1 could have gotten the object for a lower bid

### Another Game: Matching Pennies

- Each of two people chooses either Head or Tail. If the choices differ. player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.
- This is also a zero-sum or strictly competitive game
- No NE at all! What shall we do here?

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

### Conclusions

- Strategic games are one-shot games, where everybody plays its move simultaneously
- The game outcome is the action profile resulting from the individual choices.
- Each player gets a payoff based on its payoff function and the resulting action profile.
- Iterated elimination of strictly dominated strategies is a convincing solution concept, but unfortunately, most of the time it does not yield a unique solution
- Nash equilibrium is another solution concept: Action profiles, where no player has an incentive to deviate
- It also might not be unique and there can be even infinitely many NEs.
- Also, there is no guarantee for the existence of a NE