Advanced AI Techniques

I. Bayesian Networks / 1. Probabilistic Independence and Separation in Graphs

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Joint probability distributions

Pain	Y				N			
Weightloss	Y		N		Y		N	
Vomiting	Y	N	Y	N	Y	N	Y	N
Adeno Y	0.220	0.220	0.025	0.025	0.095	0.095	0.010	0.010
N	0.004	0.009	0.005	0.012	0.031	0.076	0.050	0.113

Figure 1: Joint probability distribution of four random variables P (pain), W (weightloss), V (vomiting) and A (adeno).

Marginal probability distributions

Definition 1. Let p be a the joint probability of the random variables $\mathcal{X} := \{X_1, \ldots, X_n\}$ and $\mathcal{Y} \subseteq \mathcal{X}$ a subset thereof. Then

$$p(\mathcal{Y} = y) := p^{\downarrow\mathcal{Y}}(y) := \sum_{x \in \operatorname{dom} \mathcal{X} \setminus \mathcal{Y}} p(\mathcal{X} = x, \mathcal{Y} = y)$$

is a probability distribution of $\mathcal{Y}$ called **marginal probability distribution**.

Example 1.

Vomiting	Y	N
Adeno Y	0.350	0.350
N	0.090	0.210

Pain	Y				N			
Weightloss	Y		N		Y		N	
Vomiting	Y	N	Y	N	Y	N	Y	N
Adeno Y	0.220	0.220	0.025	0.025	0.095	0.095	0.010	0.010
N	0.004	0.009	0.005	0.012	0.031	0.076	0.050	0.113

Figure 2: Joint probability distribution of four random variables P (pain), W (weightloss), V (vomiting) and A (adeno).

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Marginal probability distributions / example

all numbers in parts per 1000

	a_1	a_2	a_3	a_4
	220	330	170	280

c_3	a_1	a_2	a_3	a_4	
b_3	20	90	10	80	400
b_2	2	1	20	17	240
b_1	28	24	5	3	360

c_2	a_1	a_2	a_3	a_4
b_3	18	81	9	72
b_2	8	4	80	68
b_1	56	48	10	6

c_1	a_1	a_2	a_3	a_4
b_3	2	9	1	8
b_2	2	1	20	17
b_1	84	72	15	9

	c_1	c_2	c_3
	240	460	300

	a_1	a_2	a_3	a_4
b_3	40	180	20	160
b_2	12	6	120	102
b_1	168	144	30	18

	c_1	c_2	c_3
b_3	20	180	200
b_2	40	160	40
b_1	180	120	60

	a_1	a_2	a_3	a_4
c_3	50	115	35	100
c_2	82	133	99	146
c_1	88	82	36	34

Figure 3: Joint probability distribution and all of its marginals [BK02, p. 75].

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Conditional probability distributions

Definition 2. By $p > 0$ we mean

$$p(x) > 0, \quad \text{for all } x \in \prod \mathrm{dom}(p)$$

Then p is called **non-extreme**.

For a JPD p and a subset $\mathcal{Y} \subseteq \mathrm{dom}(p)$ of its variables with $p^{\downarrow \mathcal{Y}} > 0$ we define

$$p^{|\mathcal{Y}} := \frac{p}{p^{\downarrow \mathcal{Y}}}$$

as **conditional probability distribution of** p **w.r.t.** $\mathcal{Y}$.

A conditional probability distribution w.r.t. $\mathcal{Y}$ sums to 1 for all fixed values of $\mathcal{Y}$, i.e.,

$$(p^{|\mathcal{Y}})^{\downarrow \mathcal{Y}} \equiv 1$$

Example 2. Let p be the JPD

$$p := \begin{pmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{pmatrix}$$

on two variables R (rows) and C (columns) with the domains $\mathrm{dom}(R) = \mathrm{dom}(C) = \{1, 2\}$.

The conditional probability distribution w.r.t. C is

$$p^{|C} := \begin{pmatrix} 2/3 & 1/4 \\ 1/3 & 3/4 \end{pmatrix}$$

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Chain rule

Lemma 1 (Chain rule). *Let* $X_1, X_2, \ldots, X_n$ *be variables. Then*

$$p(X_1, X_2, \ldots, X_n) = p(X_n | X_1, \ldots, X_{n-1}) \cdots p(X_2 | X_1) \cdot p(X_1)$$

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Independent variables

Definition 3. Two sets $\mathcal{X}, \mathcal{Y}$ of variables are called **independent**, when all pairs of events $\mathcal{X} = x$ and $\mathcal{Y} = y$ are independend, i.e.

$$p(\mathcal{X} = x, \mathcal{Y} = y) = p(\mathcal{X} = x) \cdot p(\mathcal{Y} = y)$$

for all x and y or equivalently

$$p(\mathcal{X} = x | \mathcal{Y} = y) = p(\mathcal{X} = x)$$

for y with $p(\mathcal{Y} = y) > 0$.

Example 3. Let Ω be the cards in an ordinary deck and

- $R =$ true, if a card is royal,
- $T =$ true, if a card is a ten or a jack,
- $S =$ true, if a card is spade.

Cards for a single color:

2 3 4 5 6 7 8 9 10 J Q K A

ROYALS

S	R	T	$p(R, T \mid S)$
Y	Y	Y	1/13
		N	2/13
	N	Y	1/13
		N	9/13
N	Y	Y	3/39 = 1/13
		N	6/39 = 2/13
	N	Y	3/39 = 1/13
		N	27/39 = 9/13

R	T	$p(R, T)$
Y	Y	4/52 = 1/13
	N	8/52 = 2/13
N	Y	4/52 = 1/13
	N	36/52 = 9/13

Conditionally independent variables

Definition 4. Let $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ be sets of variables.

$\mathcal{X}, \mathcal{Y}$ are called **conditionally independent given** $\mathcal{Z}$, when for all events $\mathcal{Z} = z$ with $p(\mathcal{Z} = z) > 0$ all pairs of events $\mathcal{X} = x$ and $\mathcal{Y} = y$ are conditionally independend given $\mathcal{Z} = z$, i.e.

$$p(\mathcal{X} = x, \mathcal{Y} = y, \mathcal{Z} = z) = \frac{p(\mathcal{X} = x, \mathcal{Z} = z) \cdot p(\mathcal{Y} = y, \mathcal{Z} = z)}{p(\mathcal{Z} = z)}$$

for all x, y and z (with $p(\mathcal{Z} = z) > 0$), or equivalently

$$p(\mathcal{X} = x | \mathcal{Y} = y, \mathcal{Z} = z) = p(\mathcal{X} = x | \mathcal{Z} = z)$$

We write $I_p(\mathcal{X}, \mathcal{Y} | \mathcal{Z})$ for the statement, that $\mathcal{X}$ and $\mathcal{Y}$ are conditionally independent given $\mathcal{Z}$.

Conditionally independent variables

Example 4. Assume S (shape), C (color), and L (label) be three random variables that are distributed as shown in figure 4.

We show $I_p(\{L\}, \{S\}|\{C\})$, i.e., that label and shape are conditionally independent given the color.

C	S	L	$p(L\|C,S)$
black	square	1	2/6 = 1/3
		2	4/6 = 2/3
	round	1	1/3
		2	2/3
white	square	1	1/2
		2	1/2
	round	1	1/2
		2	1/2

C	L	$p(L\|C)$
black	1	3/9 = 1/3
	2	6/9 = 2/3
white	1	2/4 = 1/2
	2	2/4 = 1/2

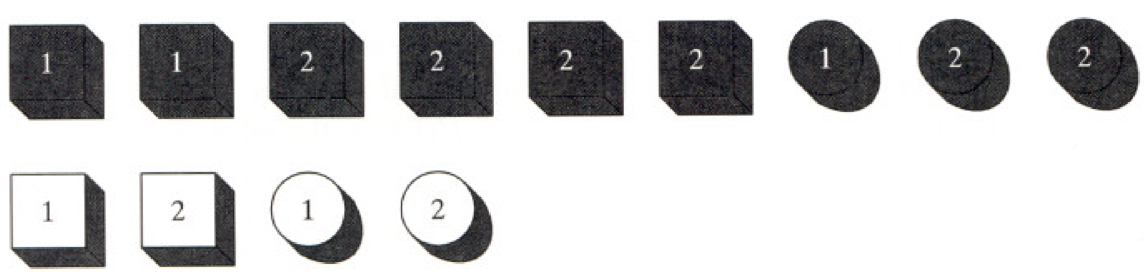

Figure 4: 13 objects with different shape, color, and label [Nea03, p. 8].

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Separation in graphs (u-separation)

Definition 5. Let $G := (V, E)$ be a graph. Let $Z \subseteq V$ be a subset of vertices. We say, two vertices $x, y \in V$ are **u-separated by Z in G**, if every path from x to y contains some vertex of Z ($\forall p \in G^* : p_1 = x, p_{|p|} = y \Rightarrow \exists i \in \{1, \ldots, n\} : p_i \in Z$).

Let $X, Y, Z \subseteq V$ be three disjoint subsets of vertices. We say, the vertices X and Y are **u-separated by Z in G**, if every path from any vertex from X to any vertex from Y is separated by Z, i.e., contains some vertex of Z.

We write $I_G(X, Y|Z)$ for the statement, that X and Y are u-separated by Z in G.

I_G is called **u-separation relation in G**.

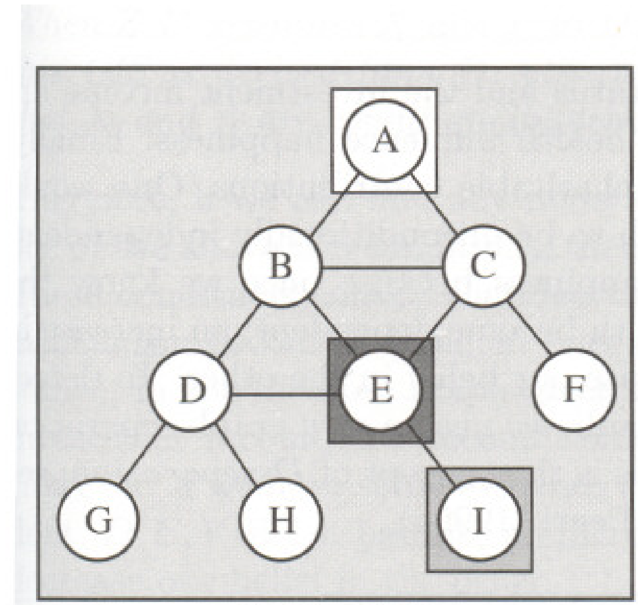

Figure 5: Example for u-separation [CGH97, p. 179].

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Separation in graphs (u-separation)

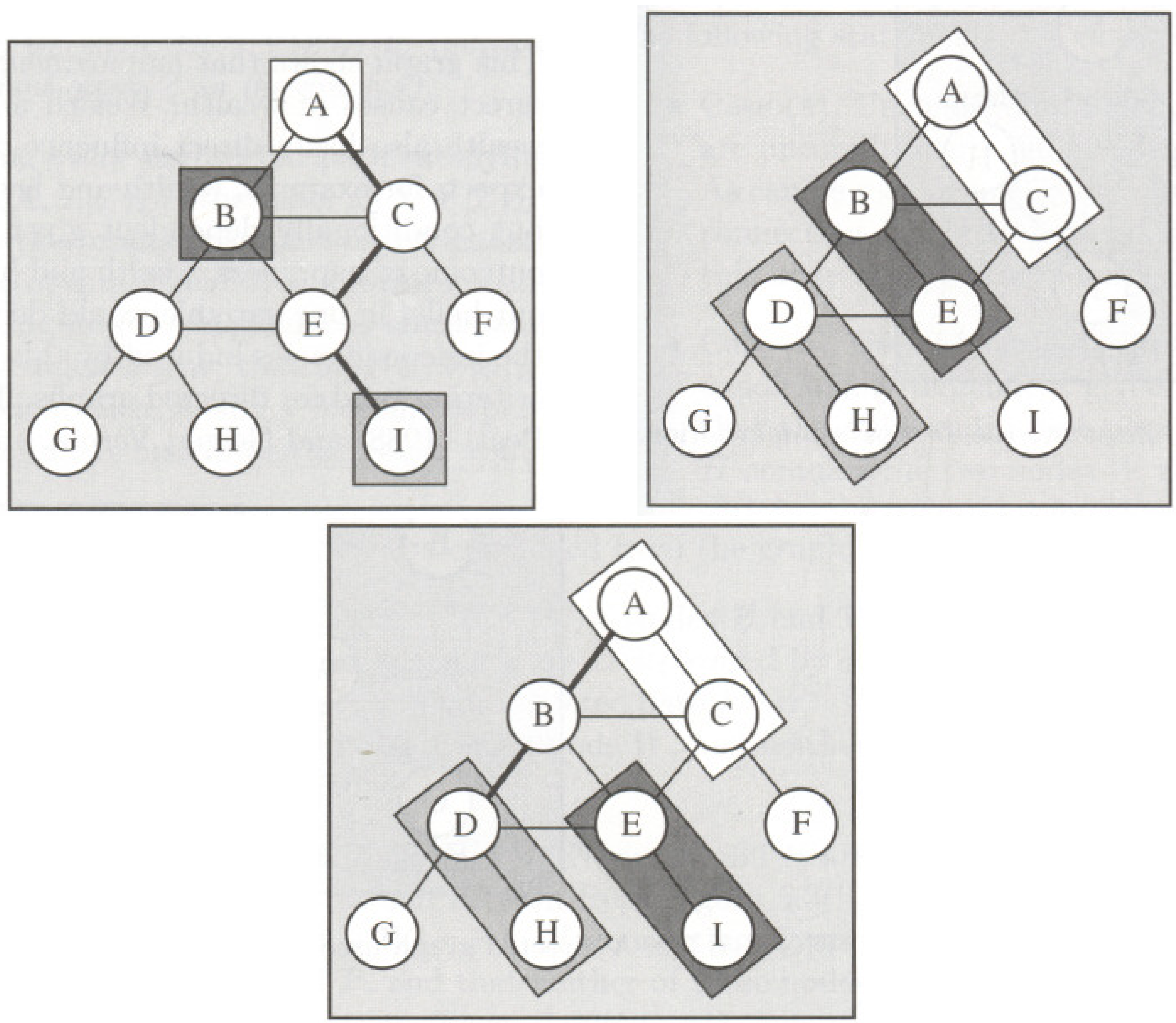

Figure 6: More examples for u-separation [CGH97, p. 179].

Checking u-separation

To test, if for a given graph $G = (V, E)$ two given sets $X, Y \subseteq V$ of vertices are u-separated by a third given set $Z \subseteq V$ of vertices, we may use standard breadth-first search to compute all vertices that can be reached from X (see, e.g., [OW02], [CLR90]).

```
breadth-first search(G, X) :
border := X
reached := ∅
while border ≠ ∅ do
        reached := reached ∪ border
        border := fan_G(border) \ reached
od
return reached
```

Figure 7: Breadth-first search algorithm for enumerating all vertices reachable from X.

For checking u-separation we have to tweak the algorithm

1. not to add vertices from Z to the border and
2. to stop if a vertex of Y has been reached.

```
check-u-separation(G, X, Y, Z) :
border := X
reached := ∅
while border ≠ ∅ do
        reached := reached ∪ border
        border := fan_G(border) \ reached \ Z
        if border ∩ Y ≠ ∅
            return false
        fi
od
return true
```

Figure 8: Breadth-first search algorithm for checking u-separation of X and Y by Z.

1. **Basic Probability Calculus**
2. **Separation in undirected graphs**
3. **Separation in directed graphs**
4. **Markov networks**
5. **Bayesian networks**

Chains

Definition 6. Let $G := (V, E)$ be a directed graph. We can construct an **undirected skeleton** $u(G) := (V, u(E))$ **of** G by dropping the directions of the edges:

$$u(E) := \{\{x, y\} \mid (x, y) \in E \text{ or } (y, x) \in E\}$$

The paths on $u(G)$ are called **chains of** G:

$$G^{\blacktriangle} := u(G)^*$$

i.e., a chain is a sequence of vertices that are linked by a forward or a backward edge. If we want to stress the directions of the linking edges, we denote a chain $p = (p_1, \dots, p_n) \in G^{\blacktriangle}$ by

$$p_1 \leftarrow p_2 \rightarrow p_3 \leftarrow \cdots \leftarrow p_{n-1} \rightarrow p_n$$

The notions of **length**, **subchain**, **interior** and **proper** carry over from undirected paths to chains.

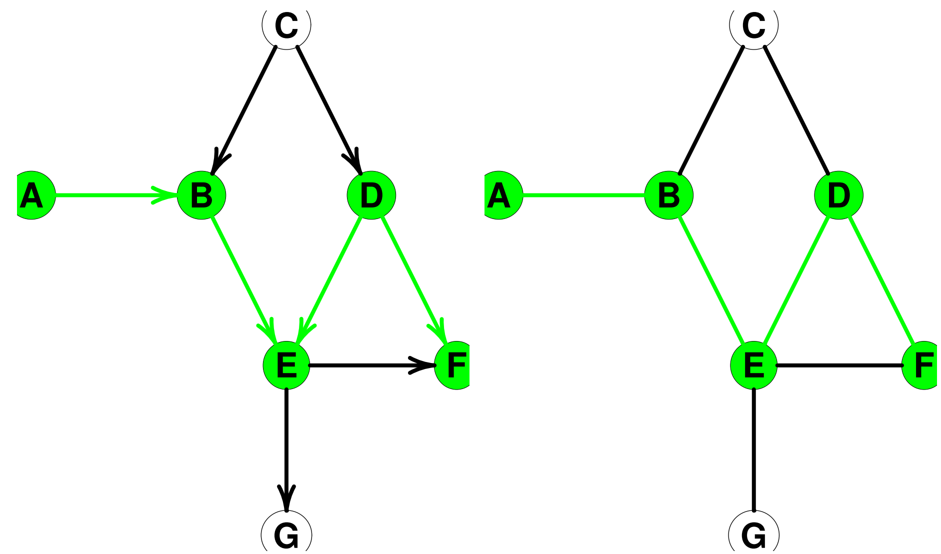

Figure 9: Chain (A, B, E, D, F) on directed graph and path on undirected skeleton.

Blocked chains

Definition 7. Let $G := (V, E)$ be a directed graph. We call a chain

$$p_1 \longrightarrow p_2 \longleftarrow p_3$$

a **head-to-head meeting**.

Let $Z \subseteq V$ be a subset of vertices.
Then a chain $p \in G^{\blacktriangle}$ is called **blocked at position i by Z**, if for its subchain (p_{i-1}, p_i, p_{i+1}) there is

$$\begin{cases} p_i \in Z, & \text{if not } p_{i-1} \longrightarrow p_i \longleftarrow p_{i+1} \\ p_i \notin Z \cup \operatorname{anc}(Z), & \text{else} \end{cases}$$

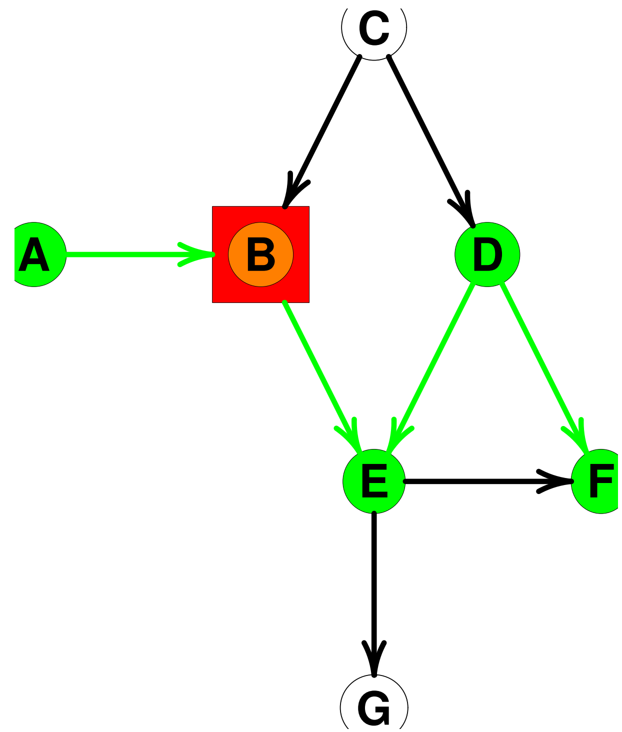

Figure 10: Chain (A, B, E, D, F) is blocked by $Z = \{B\}$ at 2.

Blocked chains / more examples

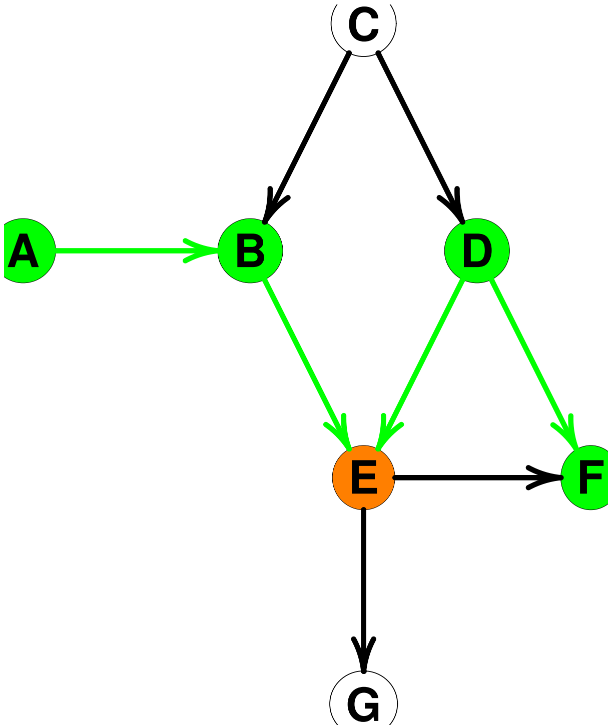

Figure 11: Chain (A, B, E, D, F) is blocked by $Z = \emptyset$ at 3.

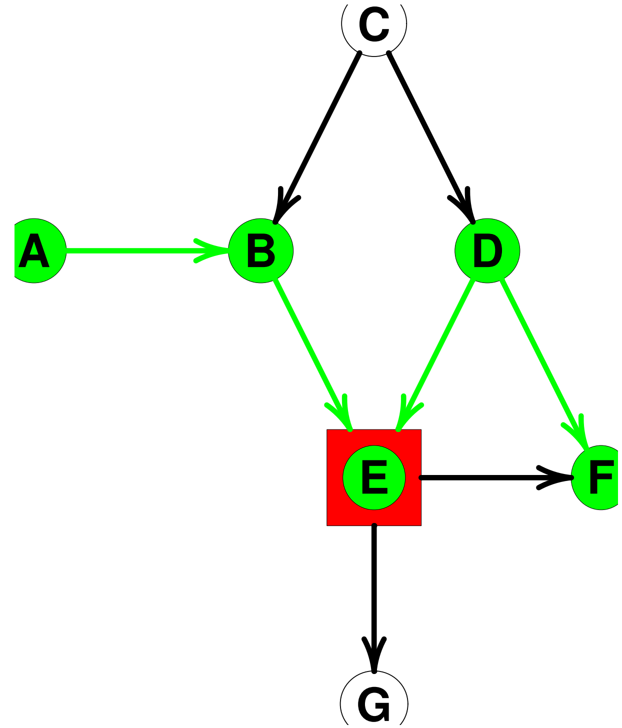

Figure 12: Chain (A, B, E, D, F) is **not** blocked by $Z = \{E\}$ at 3.

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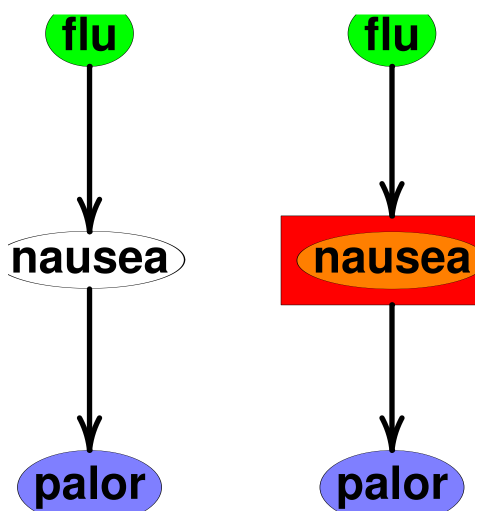
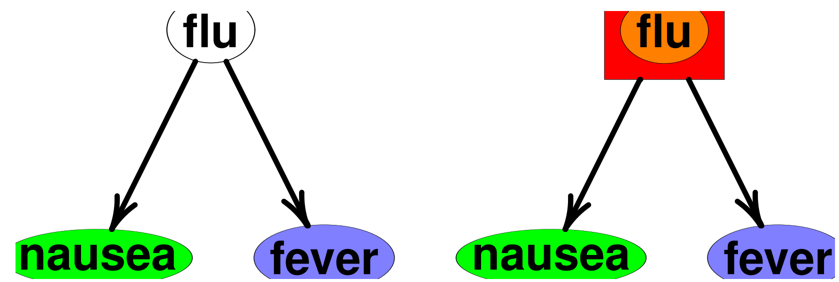
Blocked chains / rationale

The notion of blocking is choosen in a way so that chains model "flow of causal influence" through a causal network where the states of the vertices Z are already know.

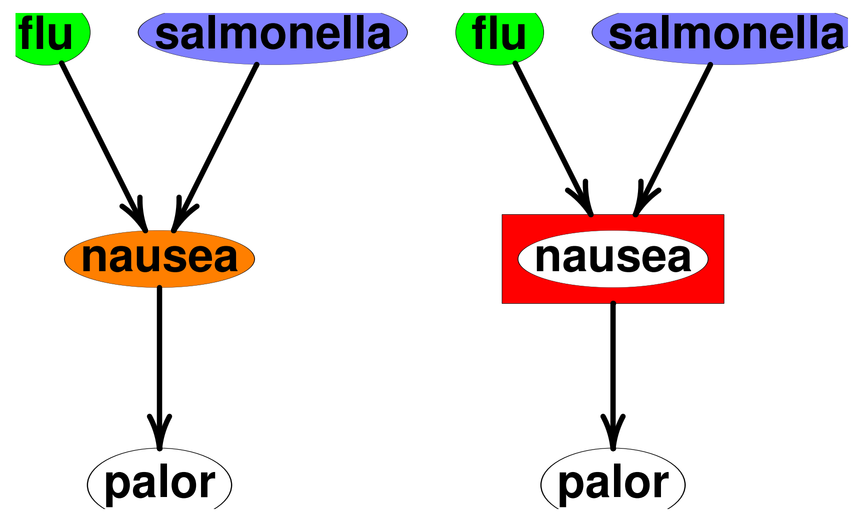
1) Serial connection / intermediate cause:



2) Diverging connection / common cause:



3) Converging connection / common effect:



Models "discounting" [Nea03, p. 51].

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The moral graph

Definition 8. Let $G := (V, E)$ be a DAG.

As the **moral graph of** G we denote the undirected skeleton graph of G plus additional edges between each two parents of a vertex, i.e. $\mathrm{moral}(G) := (V, E')$ with

$$E' := u(E) \cup \{\{x, y\} \mid \exists z \in V : x, y \in \mathrm{pa}(z)\}$$

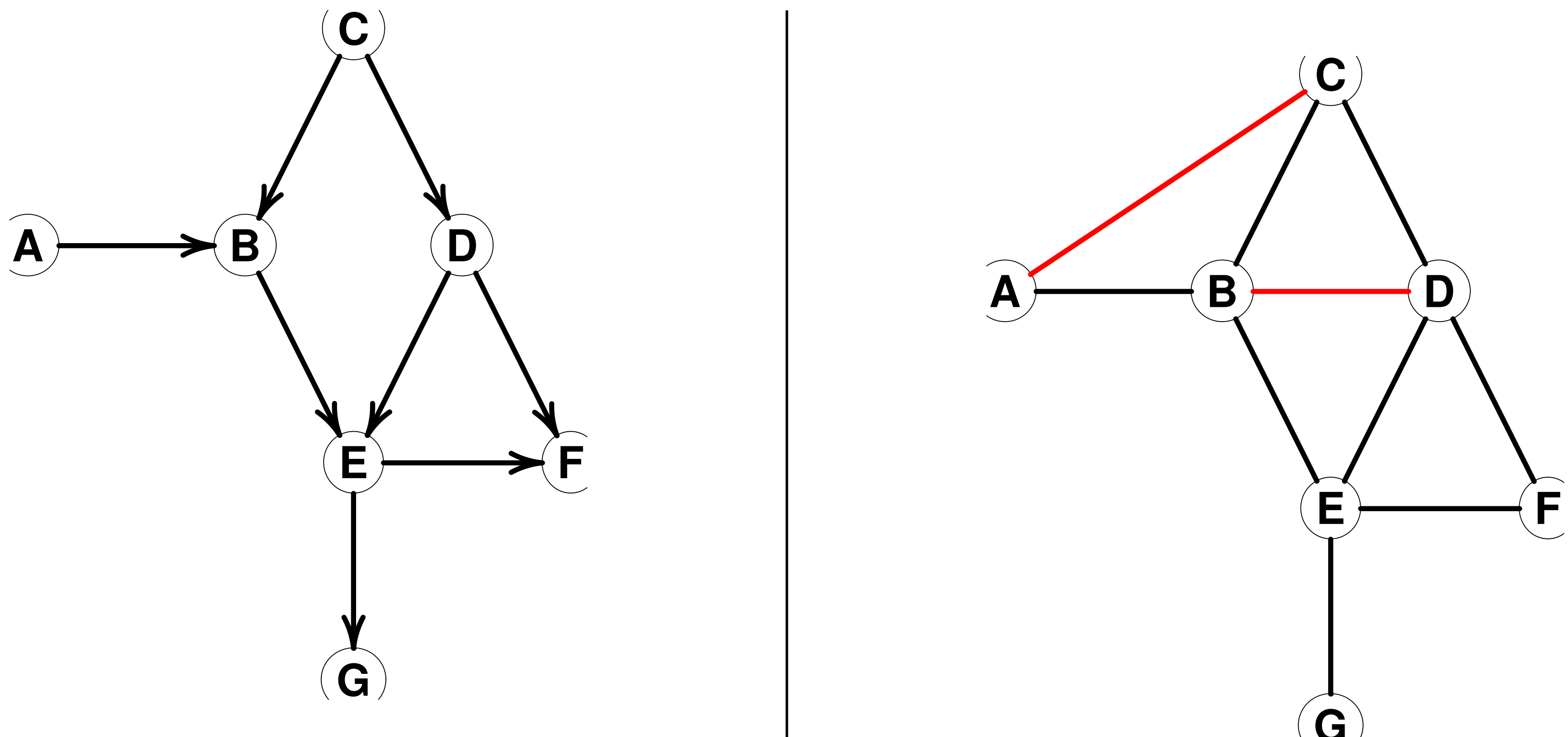

Figure 14: DAG and its moral graph.

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Separation in DAGs (d-separation)

Let $G := (V, E)$ be a DAG.

Let $X, Y, Z \subseteq V$ be three disjoint subsets of vertices. We say, the vertices X and Y are **separated by** Z **in** G, if

(i) every chain from any vertex from X to any vertex from Y is blocked by Z

or equivalently

(ii) X and Y are u-separated by Z in the moral graph of the ancestral hull of $X \cup Y \cup Z$.

We write $I_G(X, Y|Z)$ for the statement, that X and Y are separated by Z in G.

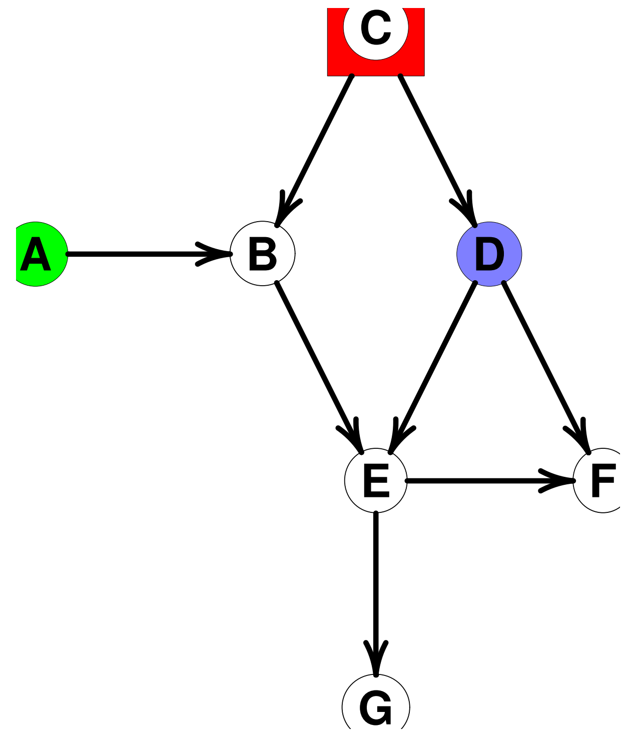

Figure 15: Are the vertices A and D separated by C in G?

Separation in DAGs (d-separation) / examples

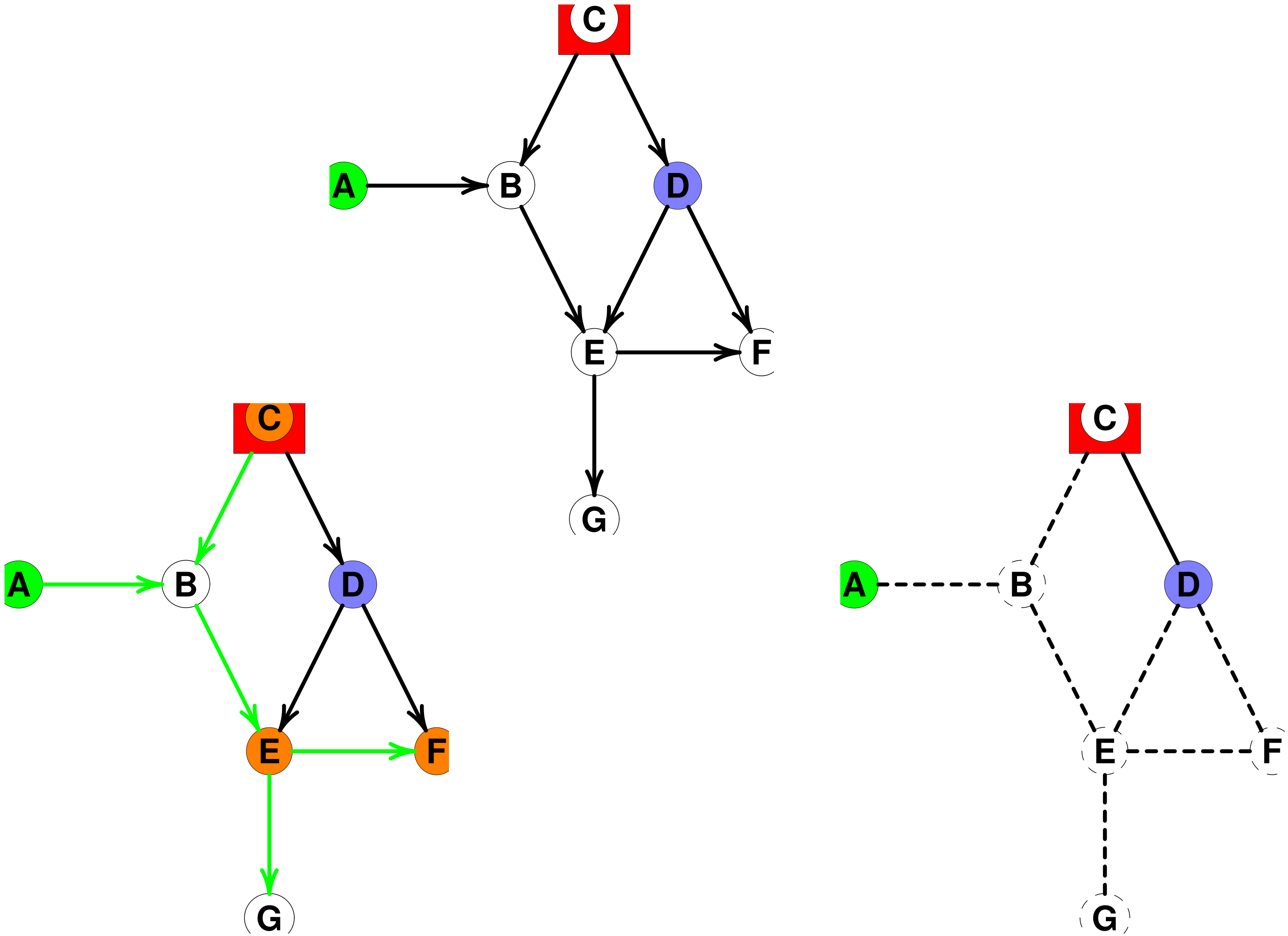

Figure 16: A and D are separated by C in G.

Separation in DAGs (d-separation) / more examples

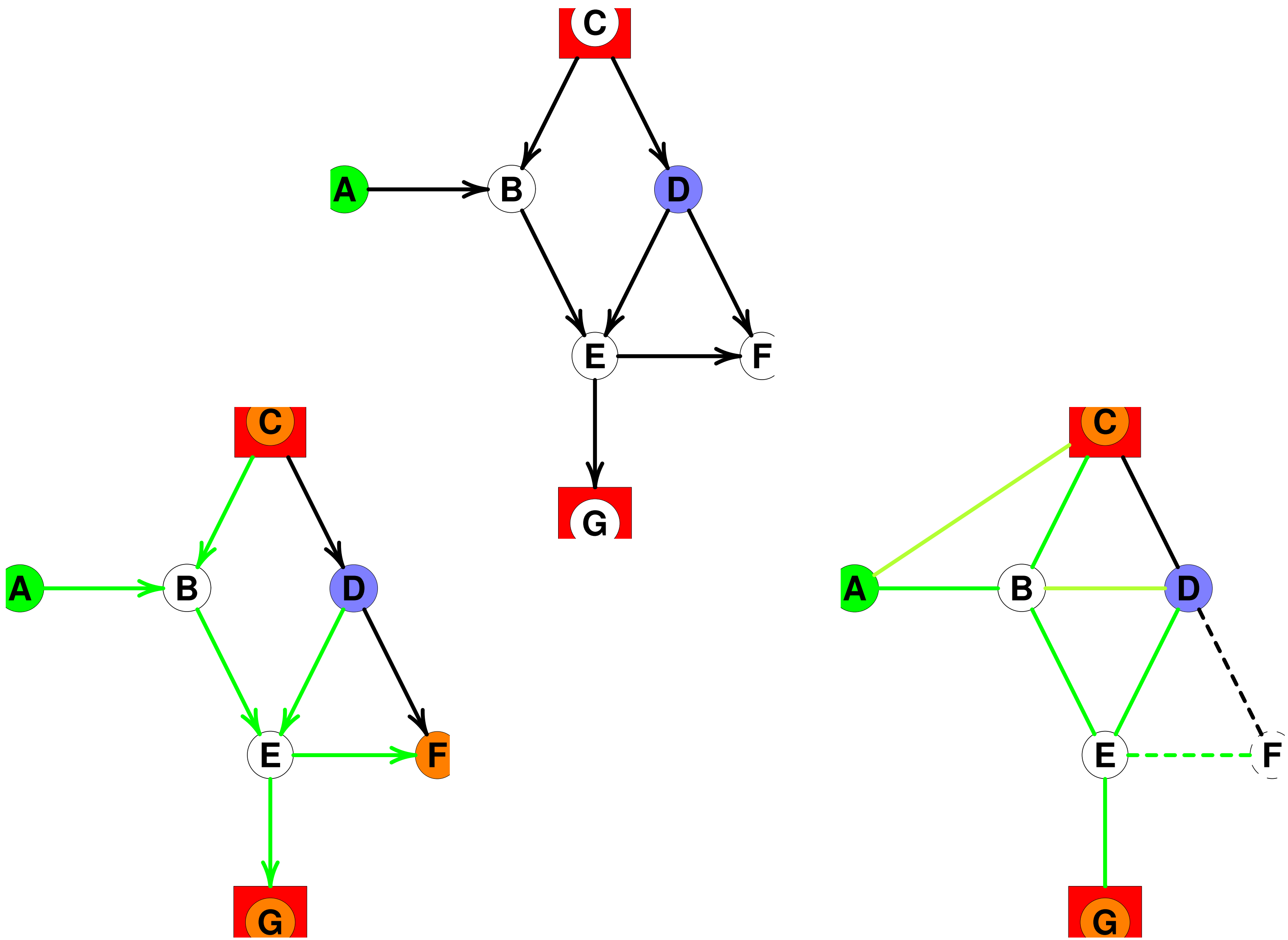

Figure 17: A and D are not separated by $\{C, G\}$ in G.

Checking d-separation

To test, if for a given graph $G = (V, E)$ two given sets $X, Y \subseteq V$ of vertices are d-separated by a third given set $Z \subseteq V$ of vertices, we may

- build the moral graph of the ancestral hull and
- apply the u-separation criterion.

$$
\begin{array}{l}
\text{check-d-separation}(G, X, Y, Z) : \\
G' := \text{moral}(\text{anc}_G(X \cup Y \cup Z)) \\
\underline{\textbf{return}}\ \textit{check-u-separation}(G', X, Y, Z)
\end{array}
$$

Figure 18: Algorithm for checking d-separation via u-separation in the moral graph.

A drawback of this algorithm is that we have to rebuild the moral graph of the ancestral hull whenever X or Y changes.

Checking d-separation

Instead of constructing a moral graph, we can modify **a breadth-first search for chains** to find all vertices not d-separated from X by Z in G.

The breadth-first search must not hop over head-to-head meetings with the middle vertex not in Z nor having an descendent in Z.

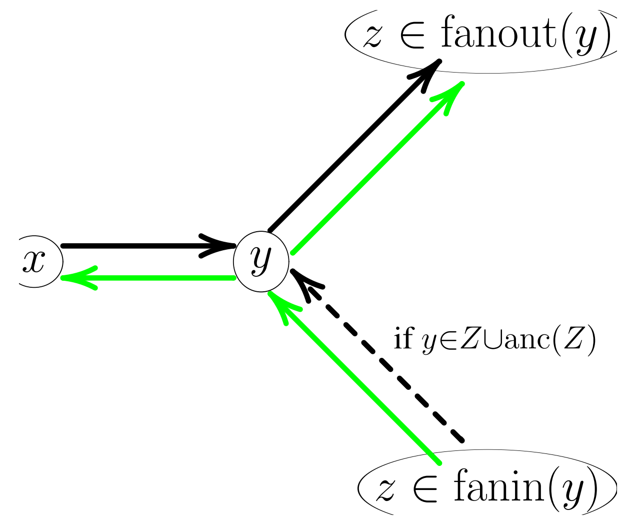

Figure 19: Restricted breadth-first search of non-blocked chains.

enumerate-d-separation$(G = (V, E), X, Z)$:
borderForward $:= \emptyset$
borderBackward $:= X \setminus Z$
reached $:= \emptyset$
while *borderForward* $\neq \emptyset$ or *borderBackward* $\neq \emptyset$ **do**
    reached $:=$ *reached* $\cup$ (*borderForward* $\setminus Z$) $\cup$ *borderBackward*
    borderForward $:= \mathrm{fanout}_G($*borderBackward* $\cup$ (*borderForward* $\setminus Z$)$) \setminus$ *reached*
    borderBackward $:= \mathrm{fanin}_G($*borderBackward* $\cup$ (*borderForward* $\cap (Z \cup \mathrm{anc}(Z))$)$) \setminus Z \setminus$ *reached*
od
return $V \setminus$ *reached*

Figure 20: Algorithm for enumerating all vertices d-separated from X by Z in G via restricted breadth-first search (see [Nea03, p. 80–86] for another formulation).

Complete graphs, orderings

Definition 9. An undirected graph $G := (V, E)$ is called **complete**, if it contains all possible edges (i.e. if $E = \mathcal{P}^2(V)$).

Definition 10. Let $G := (V, E)$ be a directed graph.
A bijective map

$$\sigma : \{1, \ldots, |V|\} \rightarrow V$$

is called an **ordering of (the vertices of)** G.

We can write an ordering as enumeration of V, i.e. as $v_1, v_2, \ldots, v_n$ with $V = \{v_1, \ldots, v_n\}$ and $v_i \neq v_j$ for $i \neq j$.

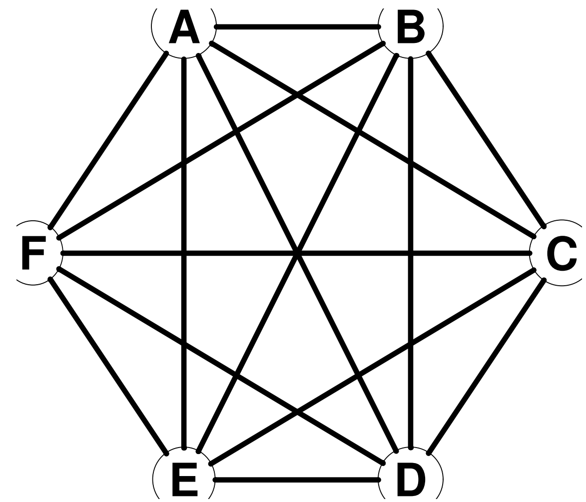

Figure 21: Undirected complete graph with 6 vertices.

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Topological orderings (1/2)

Definition 11. An ordering $\sigma = (v_1, \ldots, v_n)$ is called **topological ordering** if

(i) all parents of a vertex have smaller numbers, i.e.

$$\mathrm{fanin}(v_i) \subseteq \{v_1, \ldots, v_{i-1}\}, \quad \forall i = 1, \ldots, n$$

or equivalently

(ii) all edges point from smaller to larger numbers

$$(v, w) \in E \Rightarrow \sigma^{-1}(v) < \sigma^{-1}(w), \quad \forall v, w \in V$$

The reverse of a topological ordering – e.g. got by using the fanout instead of the fanin – is called **ancestral numbering**.

In general there are several topological orderings of a DAG.

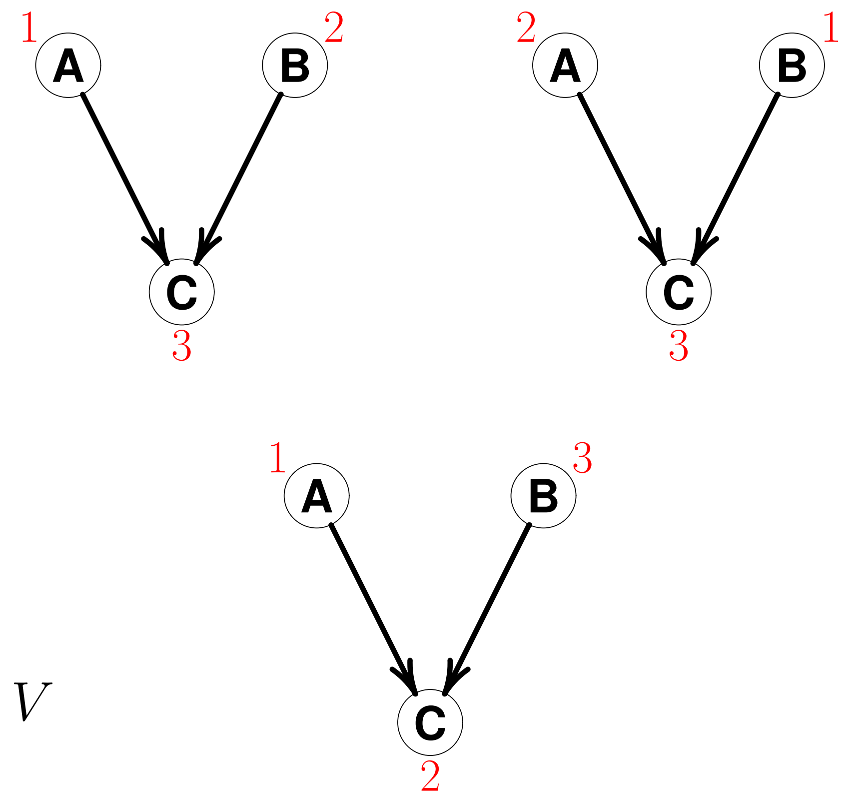

Figure 22: DAG with different topological orderings: $\sigma_1 = (A, B, C)$ and $\sigma_2 = (B, A, C)$. The ordering $\sigma_3 = (A, C, B)$ is not topological.

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Topological orderings (2/2)

Lemma 2. *Let G be a directed graph. Then*

$$G \text{ is acyclic (a DAG)} \Leftrightarrow G \text{ has a topological ordering}$$

topological-ordering$(G = (V, E))$:
choose $v \in V$ with $\mathrm{fanout}(v) = \emptyset$
$\sigma(|V|) := v$
$\sigma|_{\{1,\ldots,|V|-1\}}$:= topological-ordering$(G \setminus \{v\})$
return σ

Figure 23: Algorithm to compute a topologcial ordering of a DAG.

Exercise: write an algorithm for checking if a given directed graph is a acyclic.

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Complete DAGs

Definition 12. A DAG $G := (V, E)$ is called complete, if

(i) it has a topological ordering $\sigma = (v_1, \ldots, v_n)$ with

$$\text{fanin}(v_i) = \{v_1, \ldots, v_{i-1}\}, \quad \forall i = 1, \ldots, n$$

or equivalently

(ii) it has exactly one topological ordering

or equivalently

(iii) every additional edge introduces a cycle.

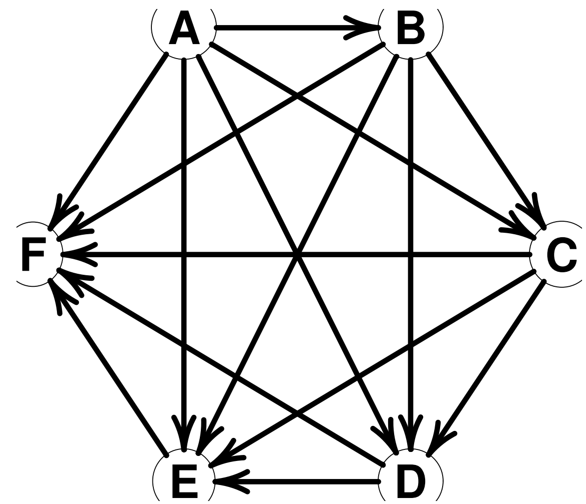

Figure 24: Complete DAG with 6 vertices. Its topological ordering is $\sigma = (A, B, C, D, E, F)$.

Graph representations of ternary relations on $\mathcal{P}(V)$

Definition 13. Let V be a set and I a ternary relation on $\mathcal{P}(V)$ (i.e. $I \subseteq \mathcal{P}(V)^3$). In our context I is often called an **independency model**.

Let G be a graph on V (undirected or DAG).
G is called a **representation of** I, if

$$I_G(X, Y|Z) \Rightarrow I(X, Y|Z) \quad \forall X, Y, Z \subseteq V$$

A representation G of I is called **faithful**, if

$$I_G(X, Y|Z) \Leftrightarrow I(X, Y|Z) \quad \forall X, Y, Z \subseteq V$$

Representations are also called **independency maps of** I or **markov w.r.t.** I, faithful representations are also called **perfect maps of** I.

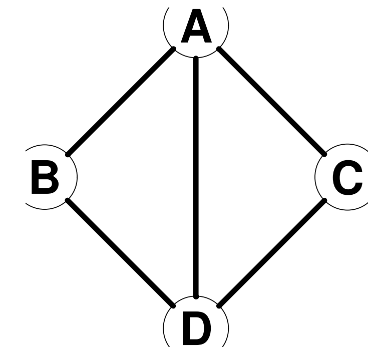

Figure 25: Non-faithful representation of

$$I := \{(A, B|\{C, D\}), (B, C|\{A, D\}), (B, A|\{C, D\}), (C, B|\{A, D\})\}$$

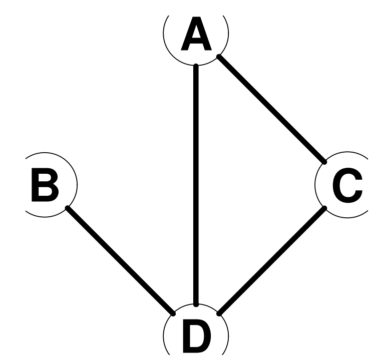

Figure 26: Faithful representation of I. Which I?

Faithful representations

In G also holds

$I_G(B, \{A, C\}|D), I_G(B, A|D), I_G(B, C|D), \ldots$

so G is not a representation of

$$I := \{(A, B|\{C, D\}), (B, C|\{A, D\}),\\ (B, A|\{C, D\}), (C, B|\{A, D\})\}$$

at all. It is a representation of

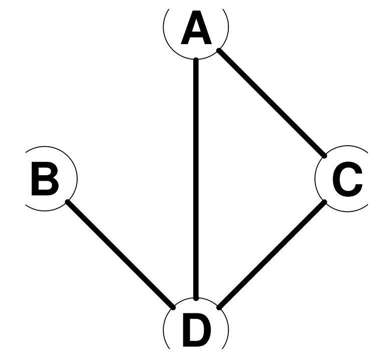

Figure 27: Faithful representation of J.

$$J := \{(A, B|\{C, D\}), (B, C|\{A, D\}), (B, \{A, C\}|D), (B, A|D), (B, C|D),\\ (B, A|\{C, D\}), (C, B|\{A, D\}), (\{A, C\}, B|D), (A, B|D), (C, B|D)\}$$

and as all independency statements of J hold in G, it is faithful.

Trivial representations

For a complete undirected graph or a complete DAG $G := (V, E)$ there is

$$I_G \equiv \text{false},$$

i.e. there are no triples $X, Y, Z \subseteq V$ with $I_G(X, Y|Z)$. Therefore G represents any independency model I on V and is called **trivial representation**.

There are independency models without faithful representation.

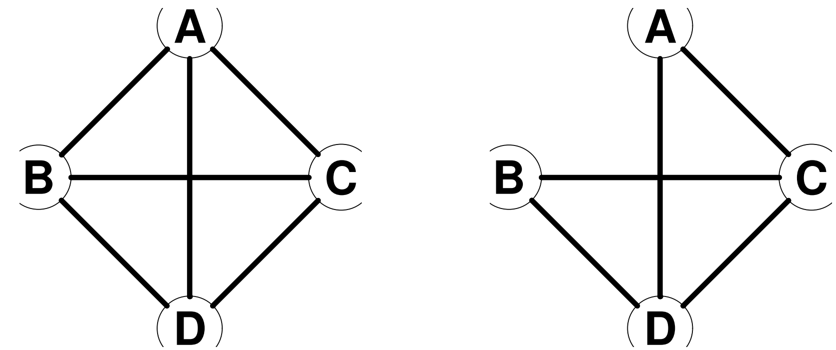

Figure 28: Independency model

$$I := \{(A, B|\{C, D\})\}$$

without faithful representation.

Minimal representations

Definition 14. A representation G of I is called **minimal**, if none of its subgraphs omitting an edge is a representation of I.

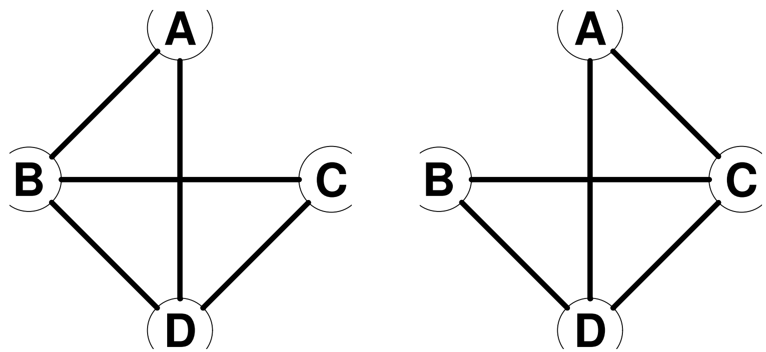

Figure 29: Different minimal undirected representations of the independency model

$$I := \{(A, B|\{C, D\}), (A, C|\{B, D\}), (B, A|\{C, D\}), (C, A|\{B, D\})\}$$

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Minimal representations

Lemma 3 (uniqueness of minimal undirected representation). *An independency model I has exactly one minimal undirected representation, if and only if it is*

(i) symmetric: $I(X, Y|Z) \Rightarrow I(Y, X|Z)$.

(ii) decomposable: $I(X, Y|Z) \Rightarrow I(X, Y'|Z) \quad$ *for any* $Y' \subseteq Y$

(iii) intersectable: $I(X, Y|Y' \cup Z)$ *and* $I(X, Y'|Y \cup Z) \Rightarrow I(X, Y \cup Y'|Z)$

Then this representation is $G = (V, E)$ *with*

$$E := \{\{x, y\} \in \mathcal{P}^2(V) \,|\, \textit{not } I(x, y|V \setminus \{x, y\}\}$$

Minimal representations (2/2)

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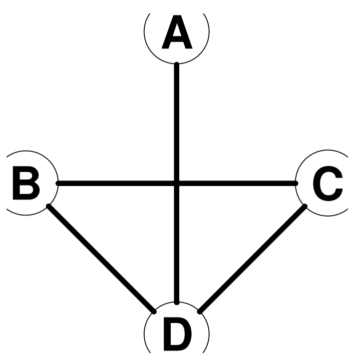
Example 5.

$$I := \{(A,B|\{C,D\}),(A,C|\{B,D\}),(A,\{B,C\}|D),(A,B|D),(A,C|D),\\ (B,A|\{C,D\}),(C,A|\{B,D\}),(\{B,C\},A|D),(B,A|D),(C,A|D)\}$$

is symmetric, decomposable and intersectable.

Its unique minimal undirected representation is



If a faithful representation exists, obviously it is the unique minimal representation, and thus can be constructed by the rule in lemma 3.

Representation of conditional independency

Definition 15. We say, a graph **represents a JPD** p, if it represents the conditional independency relation I_p of p.

General JPDs may have several minimal undirected representations (as they may violate the intersection property).

Non-extreme JPDs have a unique minimal undirected representation.

To compute this representation we have to check $I_p(X, Y | V \setminus \{X, Y\})$ for all pairs of variables $X, Y \in V$, i.e.

$$p \cdot p^{\downarrow V \setminus \{X,Y\}} = p^{\downarrow V \setminus \{X\}} \cdot p^{\downarrow V \setminus \{Y\}}$$

Then the minimal representation is the complete graph on V omitting the edges $\{X, Y\}$ for that $I_p(X, Y | V \setminus \{X, Y\})$ holds.

Representation of conditional independency

Example 6. Let p be the JPD on $V := \{X, Y, Z\}$ given by:

Z	X	Y	$p(X, Y, Z)$
0	0	0	0.024
0	0	1	0.056
0	1	0	0.036
0	1	1	0.084
1	0	0	0.096
1	0	1	0.144
1	1	0	0.224
1	1	1	0.336

Checking $p \cdot p^{\downarrow V\setminus\{X,Y\}} = p^{\downarrow V\setminus\{X\}} \cdot p^{\downarrow V\setminus\{Y\}}$ one finds that the only independency relations of p are $I_p(X, Y|Z)$ and $I_p(Y, X|Z)$.

Its marginals are:

Z	X	$p(X, Z)$
0	0	0.08
0	1	0.12
1	0	0.24
1	1	0.56

Z	Y	$p(Y, Z)$
0	0	0.06
0	1	0.14
1	0	0.32
1	1	0.48

X	Y	$p(X, Y)$
0	0	0.12
0	1	0.2
1	0	0.26
1	1	0.42

X	$p(X)$
0	0.32
1	0.68

Y	$p(Y)$
0	0.38
1	0.62

Z	$p(Z)$
0	0.2
1	0.8

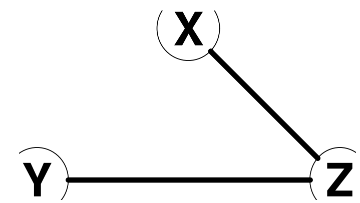
Representation of conditional independency

Example 6 (cont.).

Z	X	Y	$p(X,Y,Z)$
0	0	0	0.024
0	0	1	0.056
0	1	0	0.036
0	1	1	0.084
1	0	0	0.096
1	0	1	0.144
1	1	0	0.224
1	1	1	0.336

Checking $p \cdot p^{\downarrow V\setminus\{X,Y\}} = p^{\downarrow V\setminus\{X\}} \cdot p^{\downarrow V\setminus\{Y\}}$ one finds that the only independency relations of p are $I_p(X,Y|Z)$ and $I_p(Y,X|Z)$.

Thus, the graph



represents p, as its independency model is $I_G := \{(X,Y|Z),(Y,X|Z)\}$.

As for p only $I_p(X,Y|Z)$ and $I_p(Y,X|Z)$ hold, G is a faithful representation.

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Markov networks

Definition 16. A pair $(G, (\psi_C)_{C \in \mathcal{C}_G})$ consisting of

(i) an undirected graph G on a set of variables V and

(ii) a set of potentials

$$\psi_C : \prod_{X \in C} \mathrm{dom}(X) \to \mathbb{R}_0^+, \quad C \in \mathcal{C}_G$$

on the cliques[1)] of G (called **clique potentials**)

is called a **markov network**.

[1)] on the product of the domains of the variables of each clique.

Thus, a markov network encodes

(i) a joint probability distribution factorized as

$$p = (\prod_{C \in \mathcal{C}_G} \psi_C)^{|\emptyset}$$

and

(ii) conditional independency statements

$$I_G(X, Y|Z) \Rightarrow I_p(X, Y|Z)$$

G represents p, but not necessarily faithfully.

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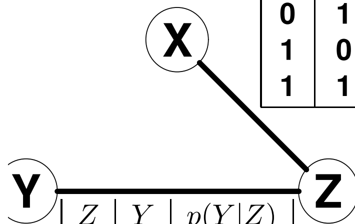
Markov networks / examples



Z	X	$p(X,Z)$
0	0	0.08
0	1	0.12
1	0	0.24
1	1	0.56

Z	Y	$p(Y\|Z)$
0	0	0.3
0	1	0.7
1	0	0.4
1	1	0.6

Figure 30: Example for a markov network.

DAG-representations

Lemma 4 (criterion for DAG-representation). *Let p be a joint probability distribution of the variables V and G be a graph on the vertices V. Then:*

G represents $p \Leftrightarrow v$ and $\text{nondesc}(v)$ are conditionally independent given $\text{pa}(v)$ for all $v \in V$, i.e.,

$$I_p(\{v\}, \text{nondesc}(v) | \text{pa}(v)), \quad \forall v \in V$$

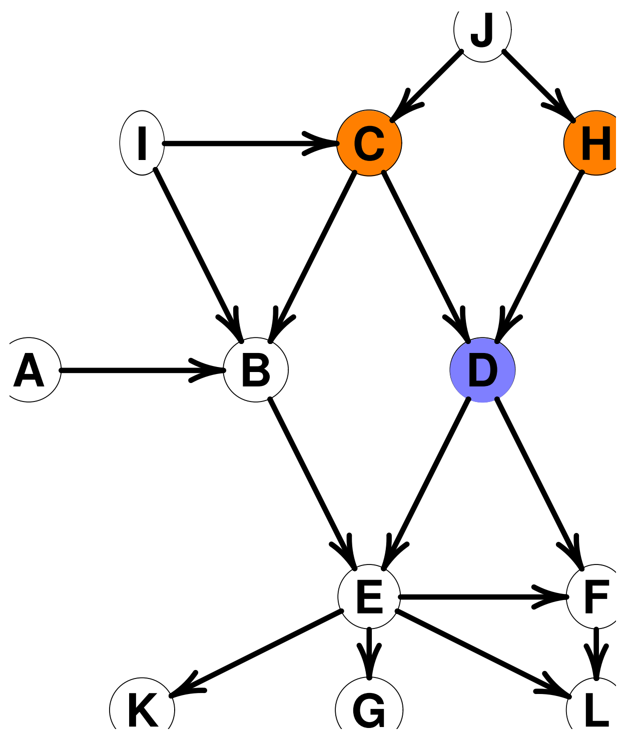


Figure 31: Parents of a vertex (orange).

Example for a not faithfully DAG-representable independency model

Probability distributions may have no faithful DAG-representation.

Example 7. The independency model

$$I := \{I(x,y|z), I(y,x|z), I(x,y|w), I(y,x|w)\}$$

does not have a faithful DAG-representation. [CGH97, p. 239]

Exercise: compute all minimal DAG-representations of I using lemma 5 and check if they are faithful.

Minimal DAG-representations

Lemma 5 (construction and uniqueness of minimal DAG-representation, [VP90]
Let I be an independence model of a JPD p. Then:

(i) A minimal DAG-representation can be constructed as follows: Choose an arbitrary ordering $\sigma := (v_1, \ldots, v_n)$ of V. Choose a minimal set $\pi_i \subseteq \{v_1, \ldots, v_{i-1}\}$ of σ-precursors of v_i with

$$I(v_i, \{v_1, \ldots, v_{i-1}\} \setminus \pi_i | \pi_i)$$

Then $G := (V, E)$ with

$$E := \{(w, v_i) \,|\, i = 1, \ldots, n, w \in \pi_i\}$$

is a minimal DAG-representation of p.

(ii) If p also is non-extreme, then the minimal representation G is unique up to ordering σ.

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Minimal DAG-representations / example

$$I := \{(A, C|B), (C, A|B)\}$$

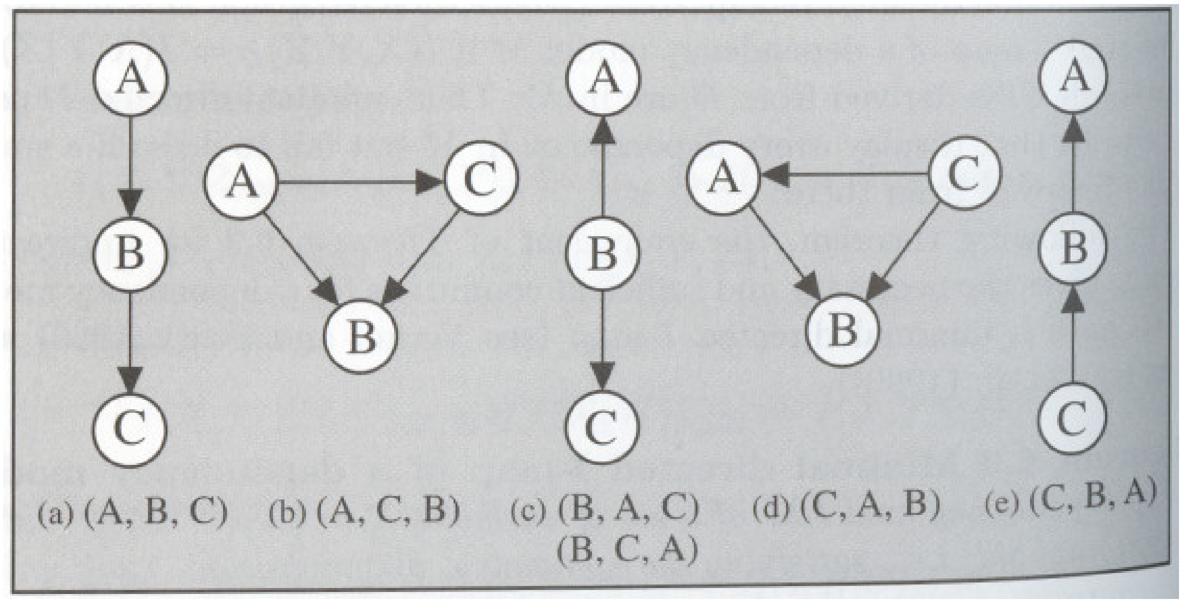

Figure 32: Minimal DAG-representations of I [CGH97, p. 240].

Minimal representations / conclusion

Representations always exist (e.g., trivial).

Minimal representations always exist
(e.g., start with trivial and drop edges successively).

	Markov network (undirected)		Bayesian network (directed)	
	minimal	faithful	minimal	faithful
general JPD	may not be unique	may not exist	may not be unique	may not exist
non-extreme JPD	unique	may not exist	unique up to ordering	may not exist

Bayesian Network

Definition 17. A pair $(G := (V, E), (p_v)_{v \in V})$ consisting of

(i) a directed graph G on a set of variables V and

(ii) a set of conditional probability distributions

$$p_X : \mathrm{dom}(X) \times \prod_{Y \in \mathrm{pa}(X)} \mathrm{dom}(Y) \to \mathbb{R}_0^+$$

at the vertices $X \in V$ conditioned on its parents (called **(conditional) vertex probability distributions**)

is called a **bayesian network**.

Thus, a bayesian network encodes

(i) a joint probability distribution factorized as

$$p = \prod_{X \in V} p(X \mid \mathrm{pa}(X))$$

and

(ii) conditional independency statements

$$I_G(X, Y|Z) \Rightarrow I_p(X, Y|Z)$$

G represents p, but not necessarily faithfully.

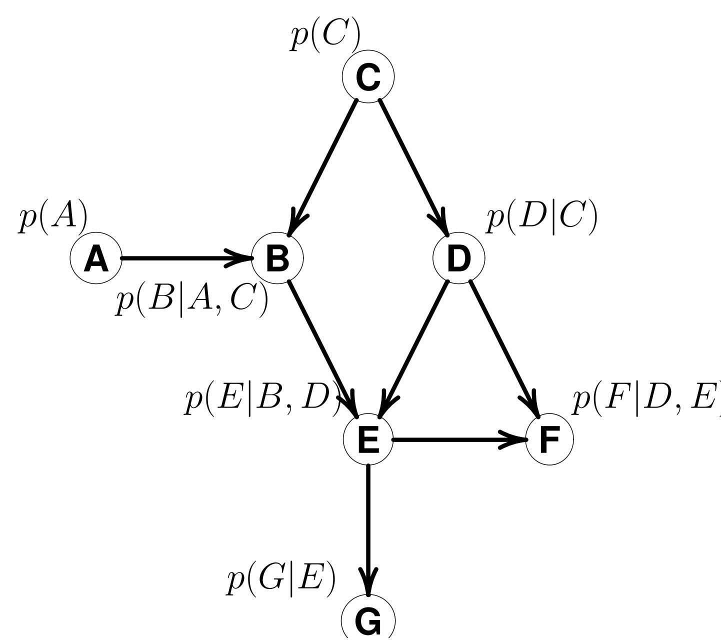

Figure 33: Example for a bayesian network.

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Types of probabilistic networks

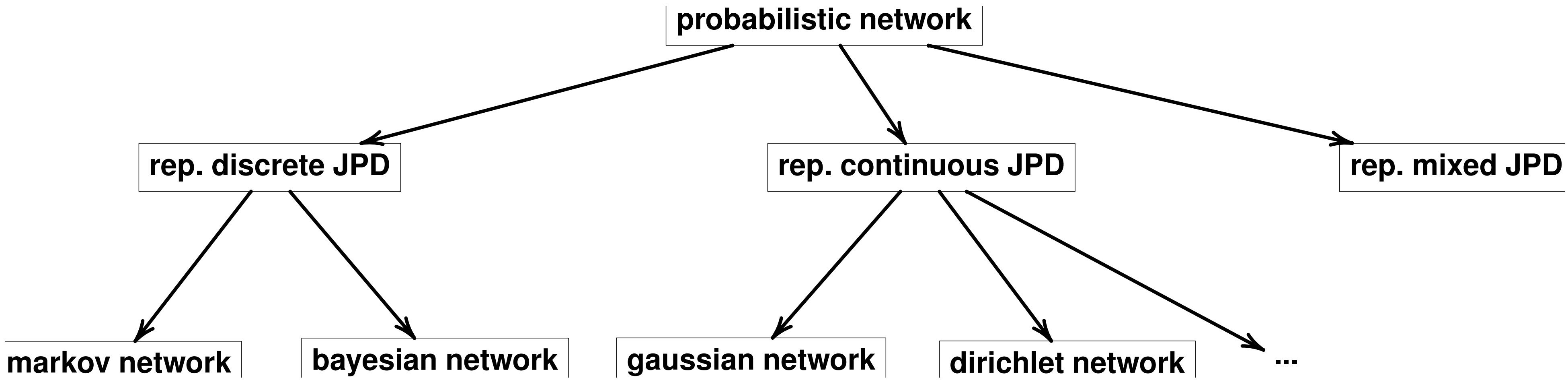

Figure 34: Types of probabilistic networks.

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References

[BK02] Christian Borgelt and Rudolf Kruse. *Graphical Models*. Wiley, New York, 2002.

[CGH97] Enrique Castillo, José Manuel Gutiérrez, and Ali S. Hadi. *Expert Systems and Probabilistic Network Models*. Springer, New York, 1997.

[CLR90] Thomas H. Cormen, Charles E. Leiserson, and Ronald L. Rivest. *Introduction to Algorithms*. MIT Press, Cambridge, Massachusetts, 1990.

[Nea03] Richard E. Neapolitan. *Learning Bayesian Networks*. Prentice Hall, 2003.

[OW02] Thomas Ottmann and Peter Widmayer. *Algorithmen und Datenstrukturen*. Spektrum Verlag, Heidelberg, 2002.

[VP90] Thomas Verma and Judea Pearl. Causal networks: semantics and expressiveness. In Ross D. Shachter, Tod S. Levitt, Laveen N. Kanal, and John F. Lemmer, editors, *Uncertainty in Artificial Intelligence 4*, pages 69–76. North-Holland, Amsterdam, 1990.