Recursive Bayes Filtering

Advanced AI

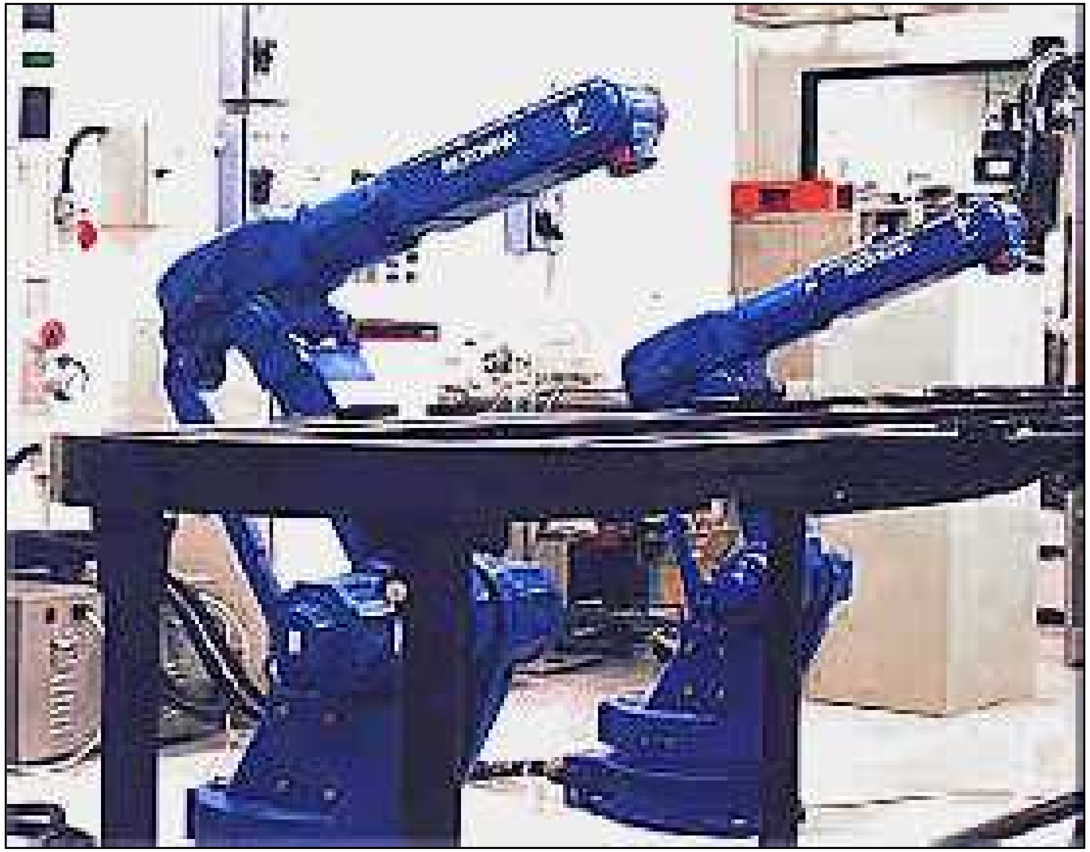
Wolfram Burgard

Tutorial Goal

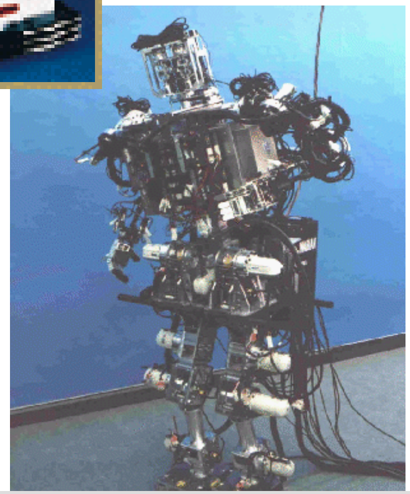
To familiarize you with
probabilistic paradigm in robotics

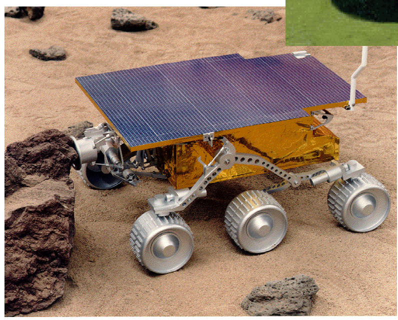
- Basic techniques
  - Advantages
  - Pitfalls and limitations
- Successful Applications
- Open research issues

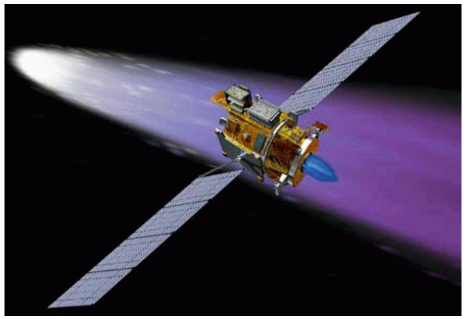
Robotics Yesterday



Robotics Today





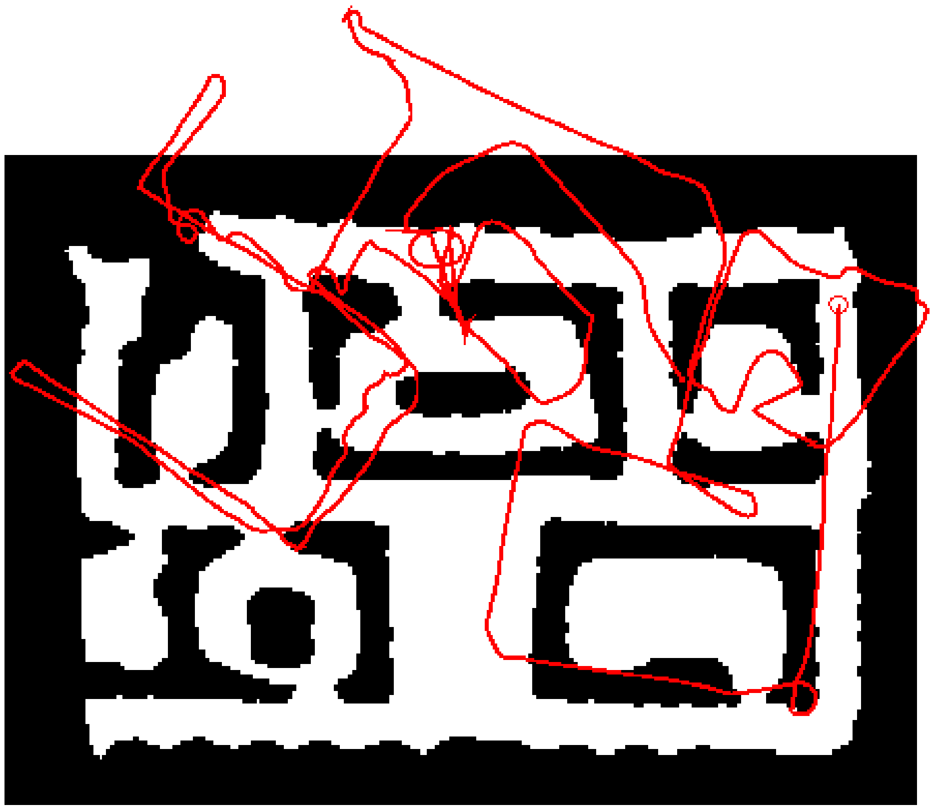


RoboCup

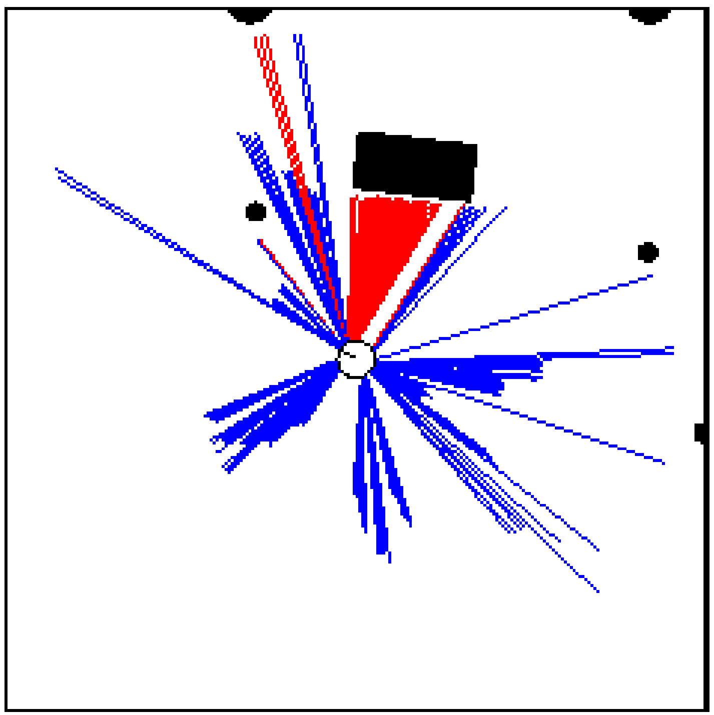
Physical Agents are Inherently Uncertain

- Uncertainty arises from four major factors:
  - Environment stochastic, unpredictable
  - Robot stochastic
  - Sensor limited, noisy
  - Models inaccurate

Nature of Sensor Data



Odometry Data



Range Data

Probabilistic Techniques for Physical Agents

Key idea: Explicit representation of uncertainty using the calculus of probability theory

Perception = state estimation

Action = utility optimization

Advantages of Probabilistic Paradigm

- Can accommodate inaccurate models
- Can accommodate imperfect sensors
- Robust in real-world applications
- Best known approach to many hard robotics problems

Pitfalls

- Computationally demanding
- False assumptions
- Approximate

Outline

- Introduction
- Probabilistic State Estimation
- Robot Localization
- Probabilistic Decision Making
  - Planning
  - Between MDPs and POMDPs
  - Exploration
- Conclusions

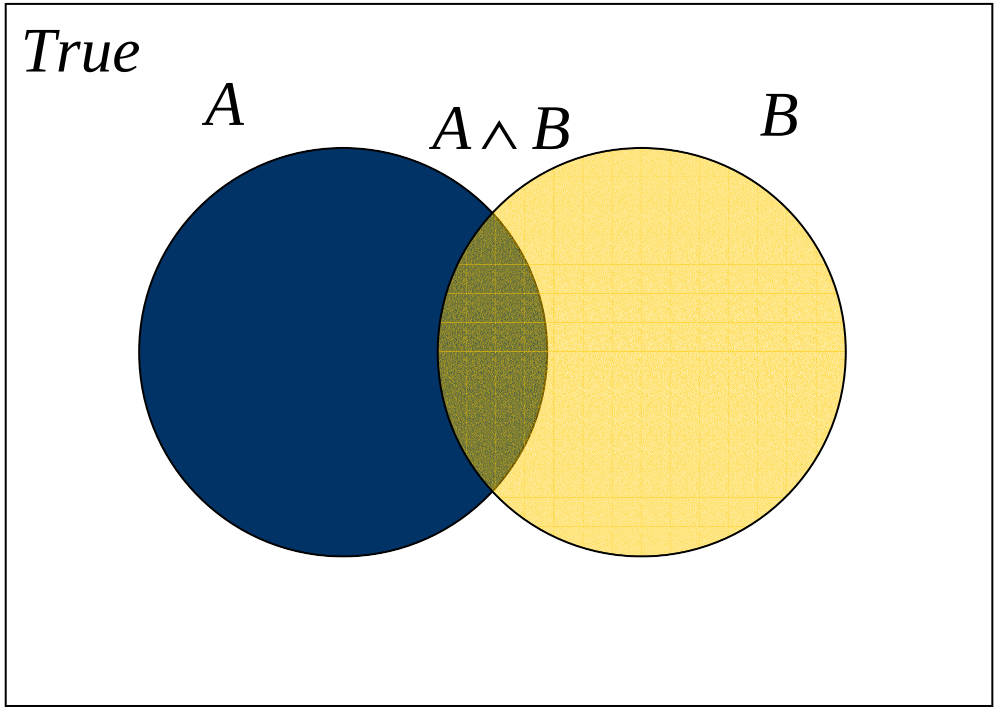
Axioms of Probability Theory

Pr*(A)* denotes probability that proposition *A* is true.

- $0 \leq \Pr(A) \leq 1$
- $\Pr(True) = 1$ $\Pr(False) = 0$
- $\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$

A Closer Look at Axiom 3

$$\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$$



Using the Axioms

$$\begin{aligned}
\Pr(A \vee \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \wedge \neg A) \\
\Pr(True) &= \Pr(A) + \Pr(\neg A) - \Pr(False) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{aligned}$$

Discrete Random Variables

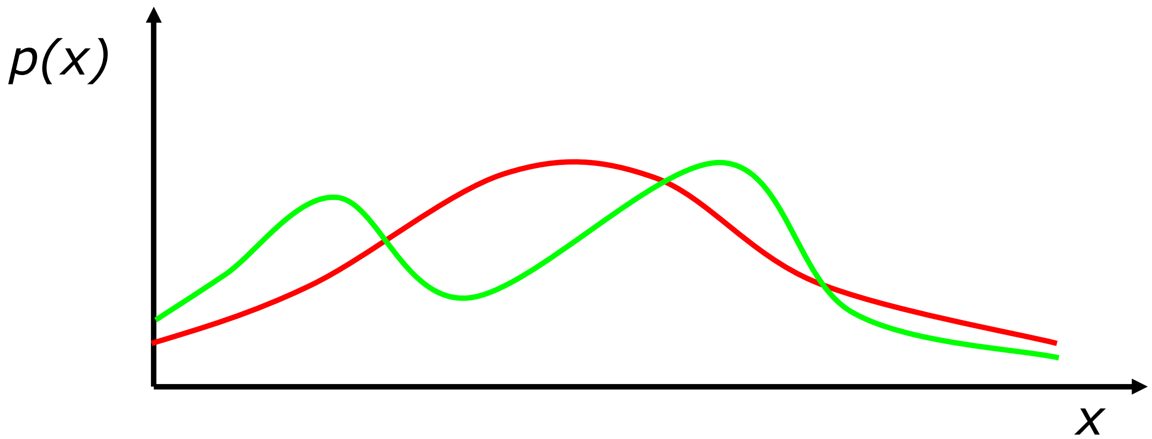
- *X* denotes a random variable.
- *X* can take on a finite number of values in $\{x_1, x_2, \ldots, x_n\}$.
- $P(X=x_i)$, or $P(x_i)$, is the probability that the random variable *X* takes on value x_i.
- $P(\cdot)$ is called probability mass function.
- E.g. $P(Room) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$

Continuous Random Variables

- X takes on values in the continuum.
- $p(X=x)$, or $p(x)$, is a probability density function.

$$\Pr(x \in [a,b]) = \int_a^b p(x)dx$$

- E.g.



Joint and Conditional Probability

- $P(X=x \text{ and } Y=y) = P(x,y)$

- If X and Y are independent then

$$P(x,y) = P(x)\, P(y)$$

- $P(x \mid y)$ is the probability of x given y

$$P(x \mid y) = P(x,y) \,/\, P(y)$$

$$P(x,y) = P(x \mid y)\, P(y)$$

- If X and Y are independent then

$$P(x \mid y) = P(x)$$

Law of Total Probability, Marginals

Discrete case

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x \mid y) P(y)$$

Continuous case

$$\int p(x)\, dx = 1$$

$$p(x) = \int p(x, y)\, dy$$

$$p(x) = \int p(x \mid y)\, p(y)\, dy$$

Bayes Formula

$$P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\Rightarrow$$

$$\boxed{P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}}$$

Normalization

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \eta\ P(y \mid x)\, P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x) P(x)}$$

Algorithm:

$$\forall x : \mathrm{aux}_{x|y} = P(y \mid x)\ P(x)$$

$$\eta = \frac{1}{\sum_x \mathrm{aux}_{x|y}}$$

$$\forall x : P(x \mid y) = \eta\ \mathrm{aux}_{x|y}$$

Conditioning

- Total probability:

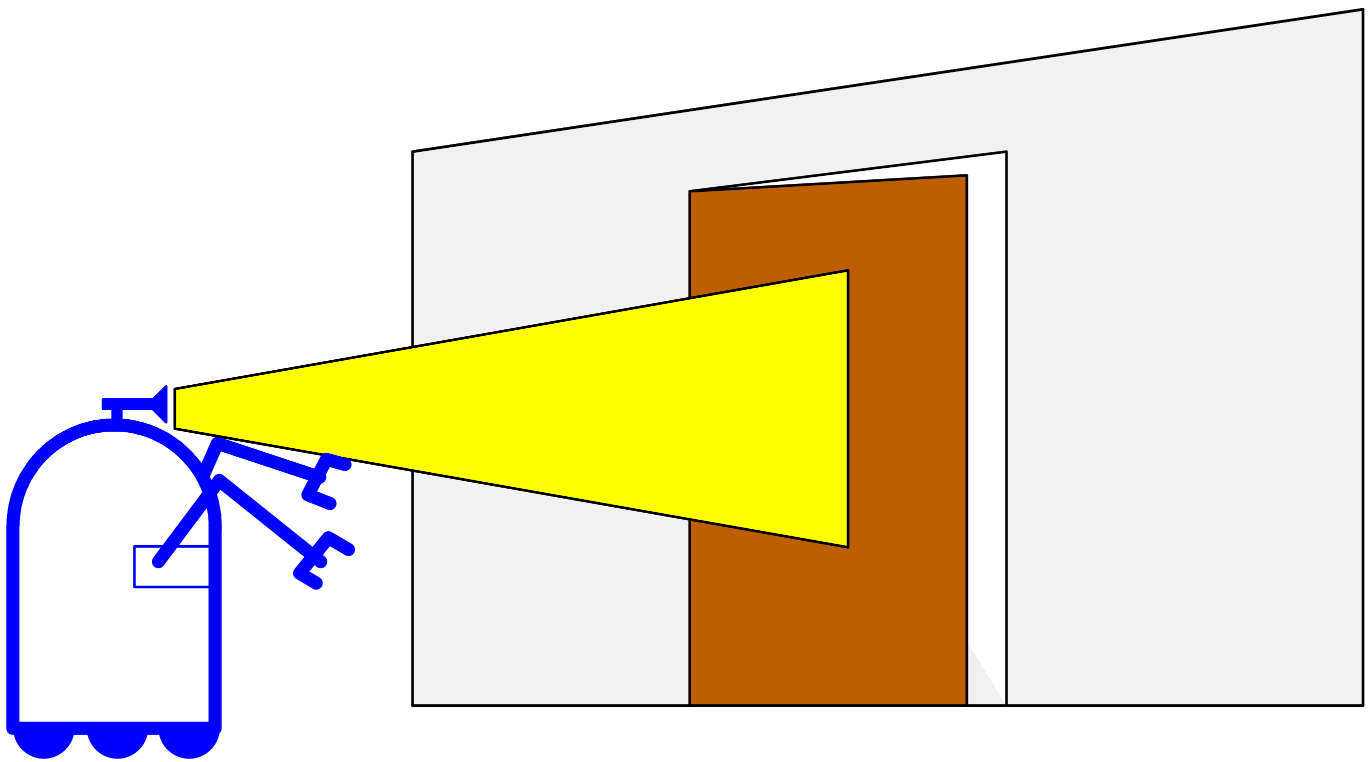
$$P(x|y) = \int P(x \mid y, z)\, P(z \mid y)\, dz$$

- Bayes rule and background knowledge:

$$P(x \mid y, z) = \frac{P(y \mid x, z)\ P(x \mid z)}{P(y \mid z)}$$

Simple Example of State Estimation

- Suppose a robot obtains measurement z
- What is *P(open|z)?*



Causal vs. Diagnostic Reasoning

- *P(open|z)* is diagnostic.
- *P(z|open)* is causal.
- Often causal knowledge is easier to obtain.
- Bayes rule allows us to use causal knowledge:

count frequencies!

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z)}$$

Example

- $P(z|open) = 0.6$ $\qquad$ $P(z|\neg open) = 0.3$
- $P(open) = P(\neg open) = 0.5$

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)\,p(open) + P(z \mid \neg open)\,p(\neg open)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67$$

- *z* raises the probability that the door is open.

Combining Evidence

- Suppose our robot obtains another observation z_2.

- How can we integrate this new information?

- More generally, how can we estimate $P(x| z_1...z_n)$?

Recursive Bayesian Updating

$$P(x \mid z_1, \dots, z_n) = \frac{P(z_n \mid x, z_1, \dots, z_{n-1})\, P(x \mid z_1, \dots, z_{n-1})}{P(z_n \mid z_1, \dots, z_{n-1})}$$

Markov assumption: z_n is independent of $z_1, \dots, z_{n-1}$ if we know x.

$$\begin{aligned} P(x \mid z_1, \dots, z_n) &= \frac{P(z_n \mid x)\, P(x \mid z_1, \dots, z_{n-1})}{P(z_n \mid z_1, \dots, z_{n-1})} \\ &= \eta\, P(z_n \mid x)\, P(x \mid z_1, \dots, z_{n-1}) \\ &= \eta_{1 \dots n} \prod_{i=1 \dots n} P(z_i \mid x)\, P(x) \end{aligned}$$

Example: Second Measurement

- $P(z_2|open) = 0.5$ $\quad\quad$ $P(z_2|\neg open) = 0.6$
- $P(open|z_1)=2/3$

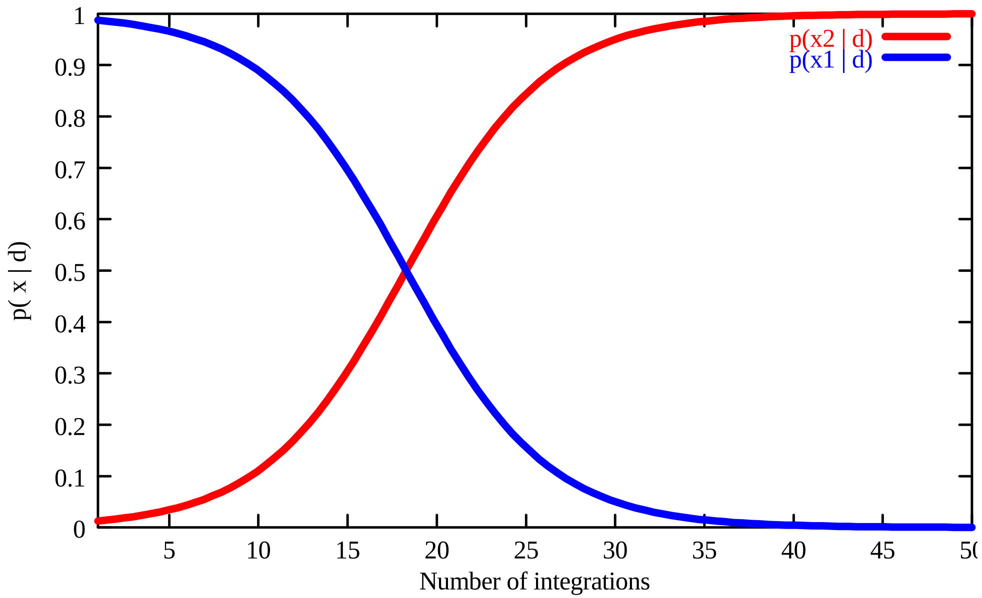
$$P(open \mid z_2, z_1) = \frac{P(z_2 \mid open)\, P(open \mid z_1)}{P(z_2 \mid open)\, P(open \mid z_1) + P(z_2 \mid \neg open)\, P(\neg open \mid z_1)}$$

$$= \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2} \cdot \frac{2}{3} + \frac{3}{5} \cdot \frac{1}{3}} = \frac{5}{8} = 0.625$$

- z_2 lowers the probability that the door is open.

A Typical Pitfall

- Two possible locations x_1 and x_2
- $P(x_1) = 1 - P(x_2) = 0.99$
- $P(z|x_2) = 0.09$ $P(z|x_1) = 0.07$



Actions

- Often the world is **dynamic** since
  - **actions carried out by the robot**,
  - **actions carried out by other agents**,
  - or just the **time** passing by

  change the world.

- How can we **incorporate** such **actions**?

Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time**…

- Actions are **never carried out with absolute certainty**.
- In contrast to measurements, **actions generally increase the uncertainty**.

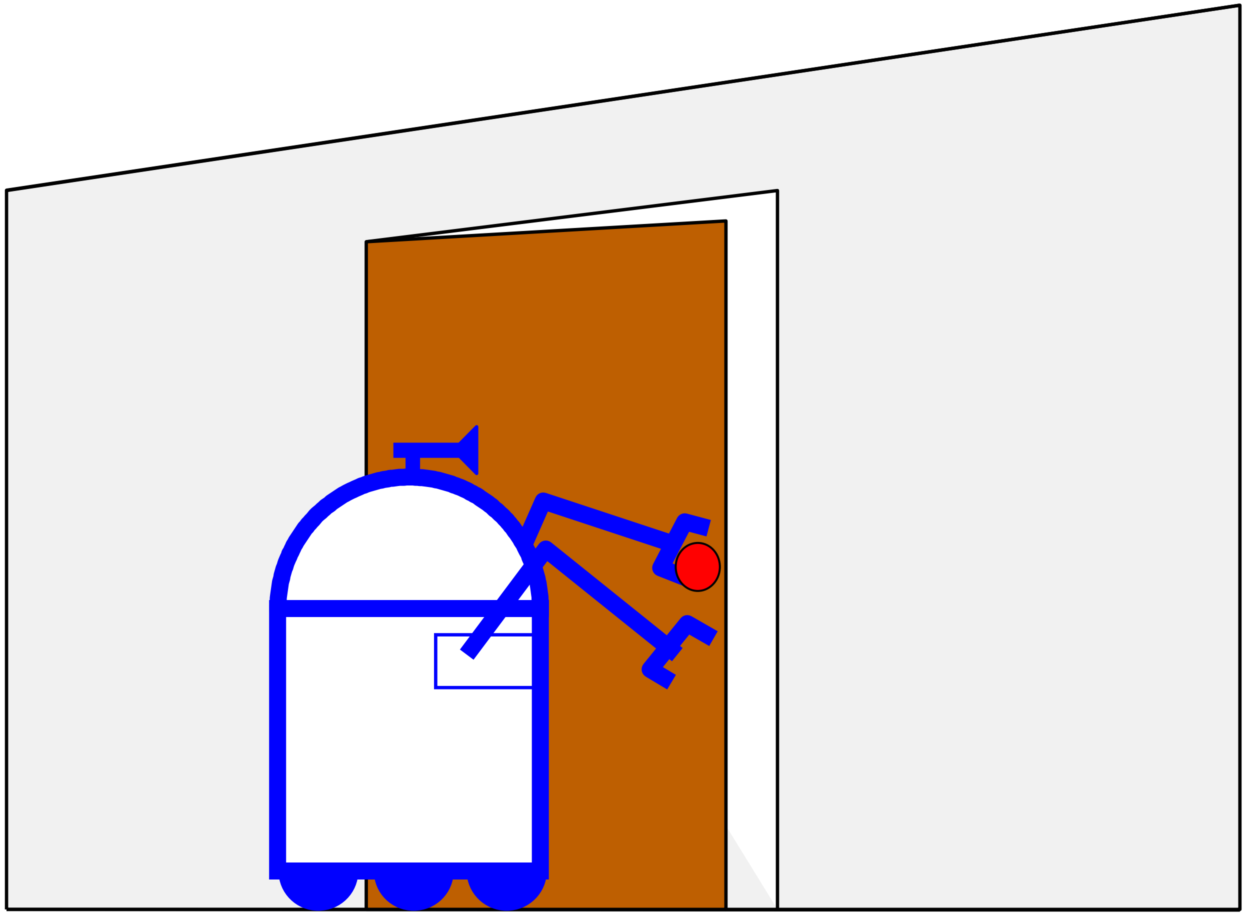
Modeling Actions

- To incorporate the outcome of an action *u* into the current "belief", we use the conditional pdf

$$P(x|u,x')$$

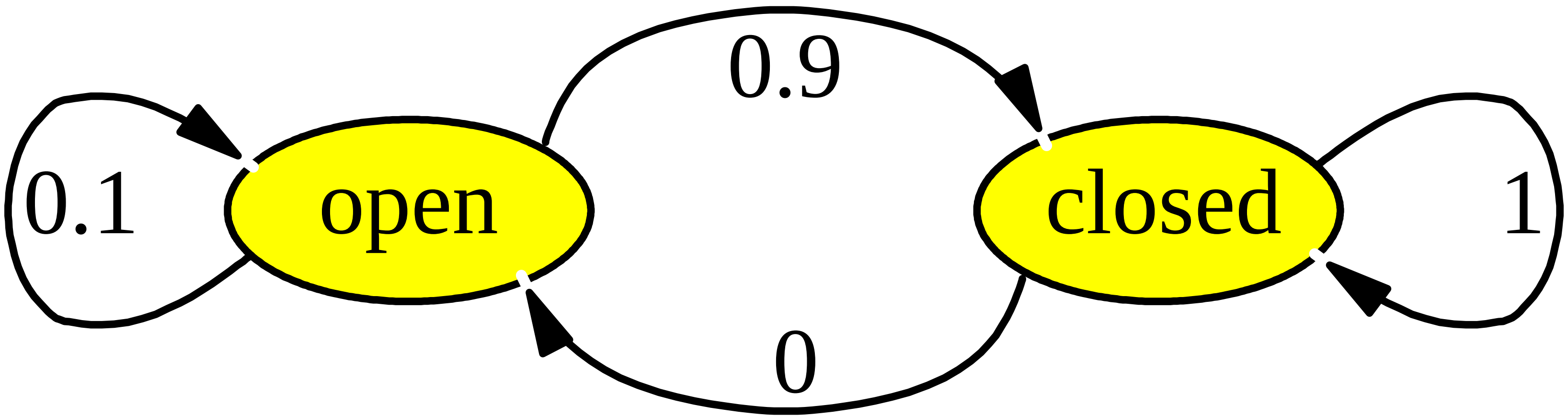
- This term specifies the pdf that **executing *u* changes the state from *x'* to *x*.**

Example: Closing the door



State Transitions

$P(x|u,x')$ for u = "close door":



If the door is open, the action "close door" succeeds in 90% of all cases.

Integrating the Outcome of Actions

Continuous case:

$$P(x \mid u) = \int P(x \mid u, x') P(x') dx'$$

Discrete case:

$$P(x \mid u) = \sum P(x \mid u, x') P(x')$$

Example: The Resulting Belief

$$
\begin{aligned}
P(closed \mid u) &= \sum P(closed \mid u, x')P(x') \\
&= P(closed \mid u, open)P(open) \\
&\quad + P(closed \mid u, closed)P(closed) \\
&= \frac{9}{10} * \frac{5}{8} + \frac{1}{1} * \frac{3}{8} = \frac{15}{16} \\
P(open \mid u) &= \sum P(open \mid u, x')P(x') \\
&= P(open \mid u, open)P(open) \\
&\quad + P(open \mid u, closed)P(closed) \\
&= \frac{1}{10} * \frac{5}{8} + \frac{0}{1} * \frac{3}{8} = \frac{1}{16} \\
&= 1 - P(closed \mid u)
\end{aligned}
$$

Bayes Filters: Framework

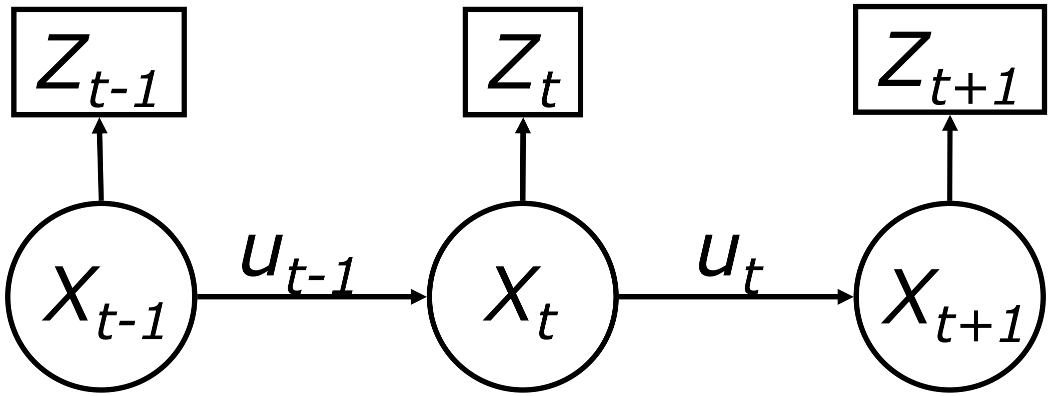
- **Given:**
  - Stream of observations *z* and action data *u:*

$$d_t = \{u_1, z_2 \ldots, u_{t-1}, z_t\}$$

  - Sensor model *P(z|x).*
  - Action model *P(x|u,x')*.
  - Prior probability of the system state *P(x).*
- **Wanted:**
  - Estimate of the state *X* of a dynamical system.
  - The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t \mid u_1, z_2 \ldots, u_{t-1}, z_t)$$

Markov Assumption



$$p(d_t, d_{t-1}, \ldots d_0 \mid x_t, d_{t+1}, d_{t+2}, \ldots) = p(d_t, d_{t-1}, \ldots, d_0 \mid x_t)$$

$$p(d_t, d_{t+1}, \ldots \mid x_t, d_1, d_2, \ldots d_{t-1}) = p(d_t, d_{t+1}, \ldots \mid x_t)$$

$$p(x_t \mid u_{t-1}, x_{t-1}, d_{t-2}, \ldots, d_0) = p(x_t \mid u_{t-1}, x_{t-1})$$

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

Bayes Filters

z = observation
u = action
x = state

$$\boxed{Bel(x_t) =} P(x_t \mid u_1, z_2 \ldots, u_{t-1}, z_t)$$

Bayes $$= \eta \; P(z_t \mid x_t, u_1, z_2, \ldots, u_{t-1}) \; P(x_t \mid u_1, z_2, \ldots, u_{t-1})$$

Markov $$= \eta \; P(z_t \mid x_t) \; P(x_t \mid u_1, z_2, \ldots, u_{t-1})$$

Total prob. $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_1, z_2, \ldots, u_{t-1}, x_{t-1}) \; P(x_{t-1} \mid u_1, z_2, \ldots, u_{t-1}) \; dx_{t-1}$$

Markov $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_{t-1}, x_{t-1}) \; P(x_{t-1} \mid u_1, z_2, \ldots, u_{t-1}) \; dx_{t-1}$$

$$= \boxed{\eta \; P(z_t \mid x_t) \int P(x_t \mid u_{t-1}, x_{t-1}) \; Bel(x_{t-1}) \; dx_{t-1}}$$

$$Bel(x_t) = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_{t-1}, x_{t-1}) \ Bel(x_{t-1}) \ dx_{t-1}$$

1. Algorithm **Bayes_filter**(*Bel(x),d*):
2. η=0
3. if *d* is a perceptual data item *z* then
4. For all *x* do
5. $Bel'(x) = P(z \mid x) Bel(x)$
6. $\eta = \eta + Bel'(x)$
7. For all *x* do
8. $Bel'(x) = \eta^{-1} Bel'(x)$
9. else if *d* is an action data item *u* then
10. For all *x* do
11. $Bel'(x) = \int P(x \mid u, x') \ Bel(x') \ dx'$
12. return *Bel′(x)*

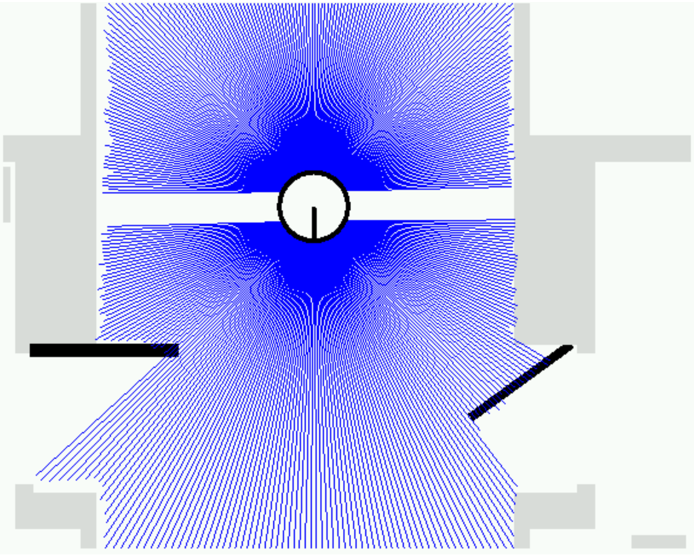
Bayes Filters are Familiar!

$$Bel(x_t) = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_{t-1}, x_{t-1}) \ Bel(x_{t-1}) \ dx_{t-1}$$

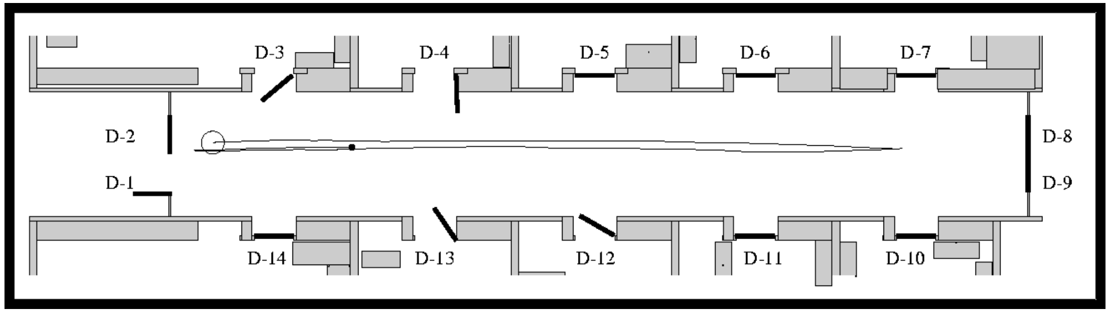
- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayes networks
- Partially Observable Markov Decision Processes (POMDPs)

Application to Door State Estimation

- Estimate the opening angle of a door
- and the state of other dynamic objects
- using a laser-range finder
- from a moving mobile robot and
- based on Bayes filters.



Result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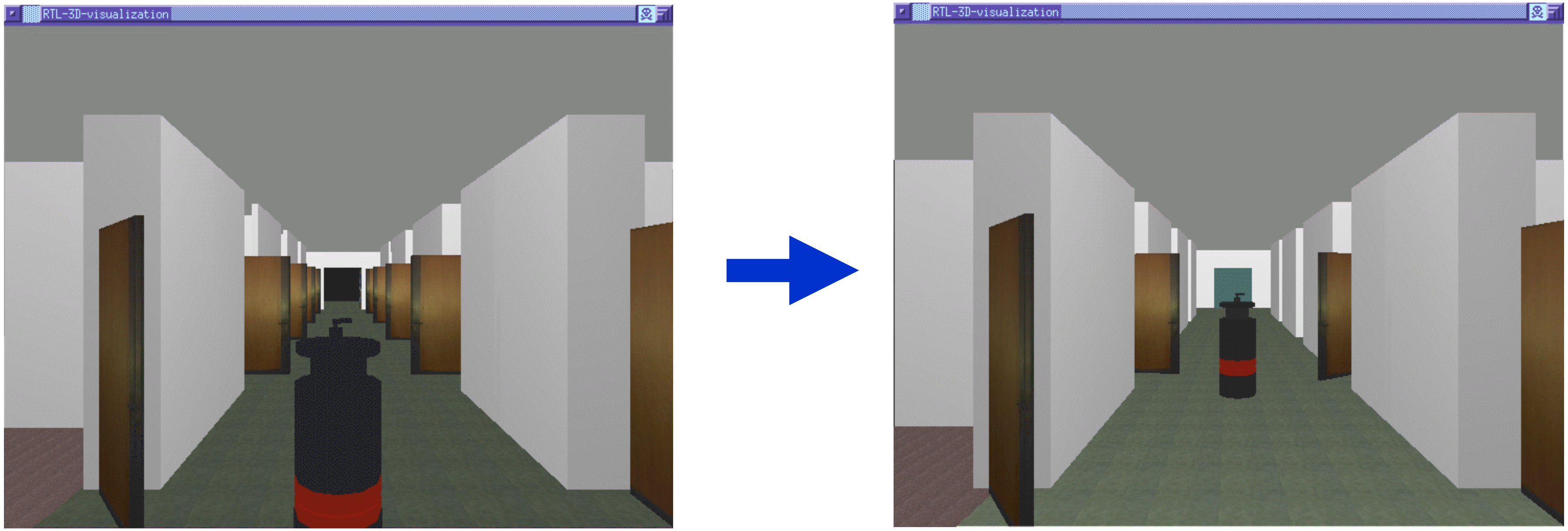

RTL-3D-visualization
RTL-3D-visualization

Lessons Learned

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.

Tutorial Outline

- Introduction
- Probabilistic State Estimation
- Localization
- Probabilistic Decision Making
  - Planning
  - Between MDPs and POMDPs
  - Exploration
- Conclusions

The Localization Problem

> "Using sensory information to locate the robot in its environment is the most fundamental problem to providing a mobile robot with autonomous capabilities." [Cox '91]

- **Given**
  - Map of the environment.
  - Sequence of sensor measurements.
- **Wanted**
  - Estimate of the robot's position.
- **Problem classes**
  - Position tracking
  - Global localization
  - Kidnapped robot problem (recovery)

Representations for Bayesian Robot Localization

Discrete approaches ('95)
- Topological representation ('95)
  - uncertainty handling (POMDPs)
  - occas. global localization, recovery
- Grid-based, metric representation ('96)
  - global localization, recovery

Particle filters ('99)
- sample-based representation
- global localization, recovery

AI

Kalman filters (late-80s?)
- Gaussians
- approximately linear models
- position tracking

Multi-hypothesis ('00)
- multiple Kalman filters
- global localization, recovery

Robotics

What is the Right Representation?

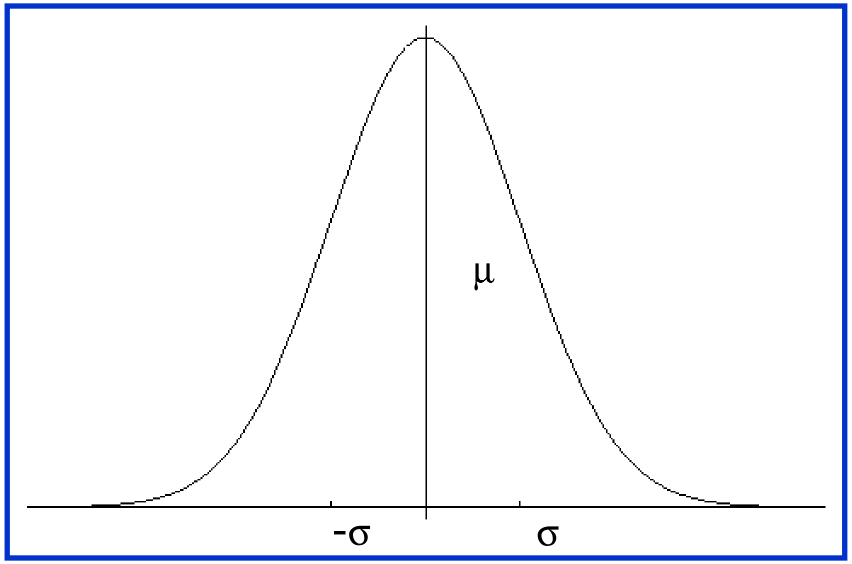
- Kalman filters
- Multi-hypothesis tracking
- Grid-based representations
- Topological approaches
- Particle filters

Gaussians

$$p(x) \sim N(\mu, \sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

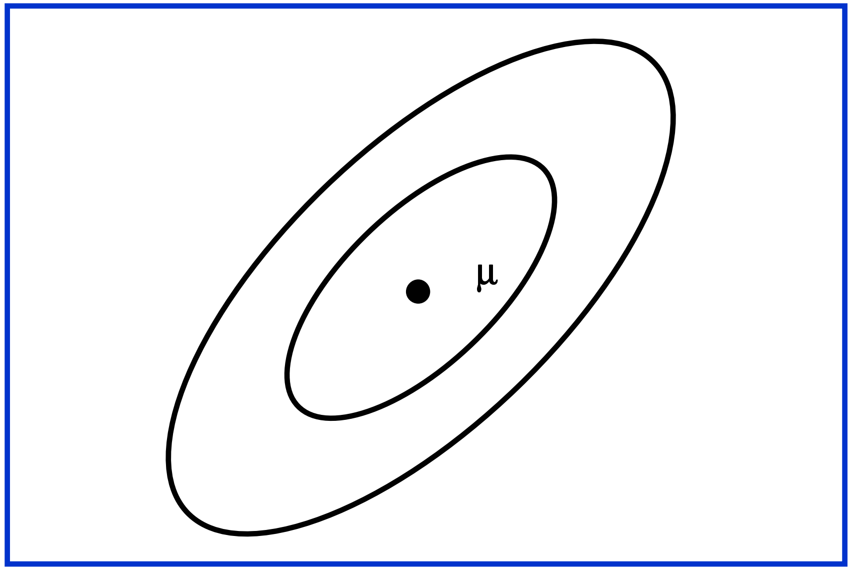
Univariate



$$p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}):$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$

Multivariate



Kalman Filters

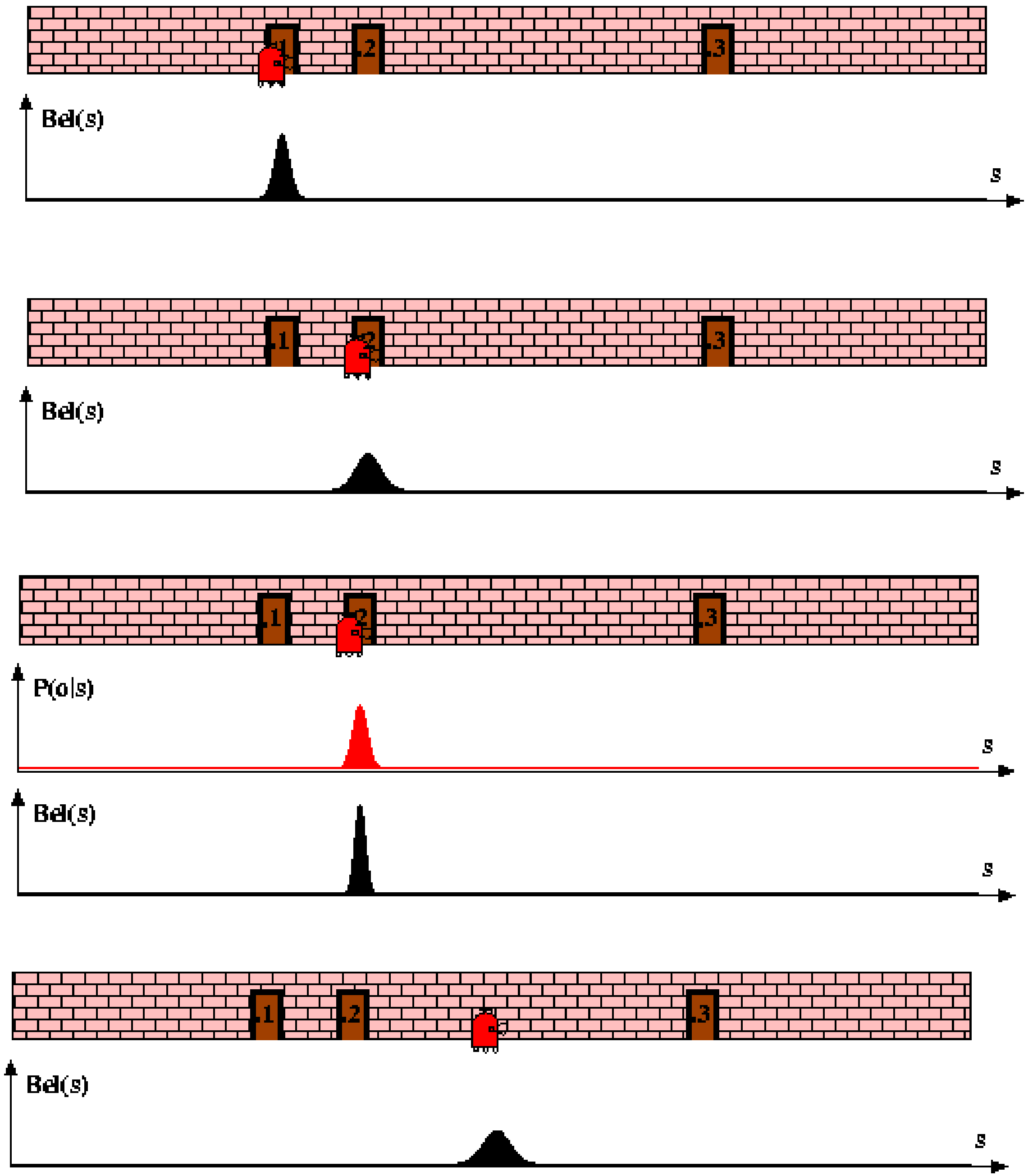
Estimate the state of processes that are governed by the following linear stochastic difference equation.

$$x_{t+1} = Ax_t + Bu_t + v_t$$

$$z_t = Cx_t + w_t$$

The random variables v_t and w_t represent the process measurement noise and are assumed to be independent, white and with normal probability distributions.

Kalman Filters



[Schiele et al. 94], [Weiß et al. 94], [Borenstein 96], [Gutmann et al. 96, 98], [Arras 98]

Kalman Filter Algorithm

1. Algorithm **Kalman_filter**(<μ,Σ>, *d*):

2. If *d* is a perceptual data item *z* then
3. $K = \Sigma C^T \left(C \Sigma C^T + \Sigma_{obs} \right)^{-1}$
4. $\mu = \mu + K(z - C\mu)$
5. $\Sigma = (I - KC)\Sigma$

6. Else if *d* is an action data item *u* then
7. $\mu = A\mu + Bu$
8. $\Sigma = A \Sigma A^T + \Sigma_{act}$

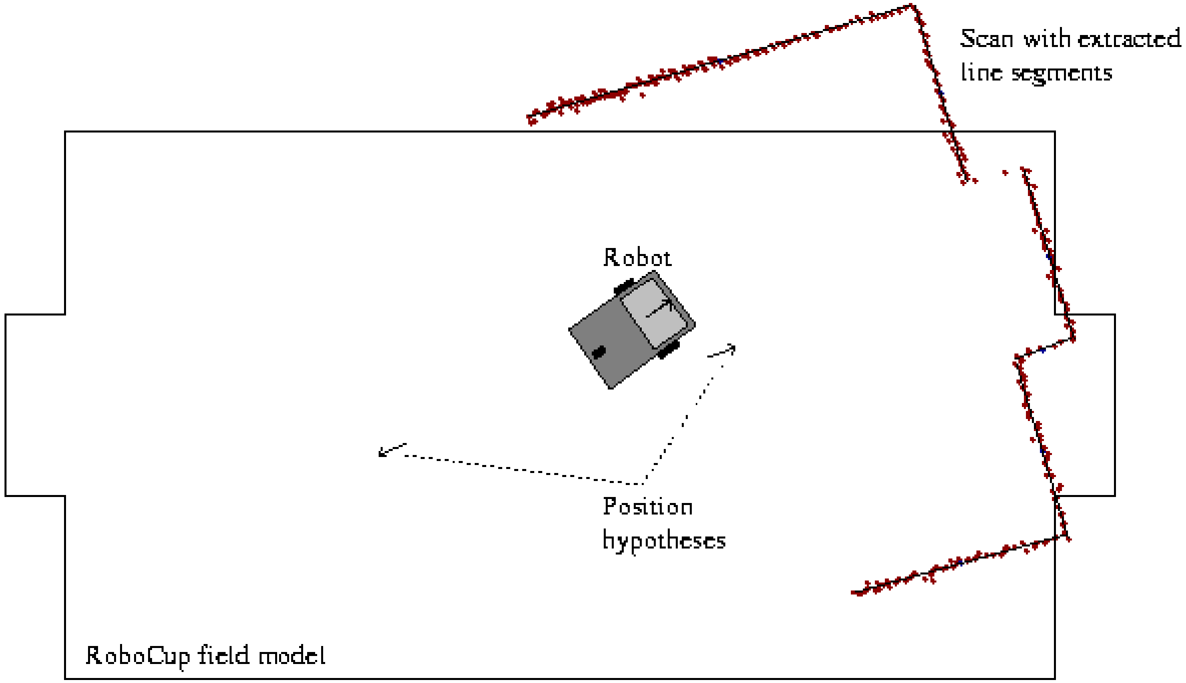
9. Return <μ,Σ>

Non-linear Systems

- Very strong assumptions:
  - Linear state dynamics
  - Observations linear in state
- What can we do if system is not linear?
  - Linearize it: **EKF**
  - Compute the Jacobians of the dynamics and observations at the current state.
  - Extended Kalman filter works surprisingly well even for highly non-linear systems.

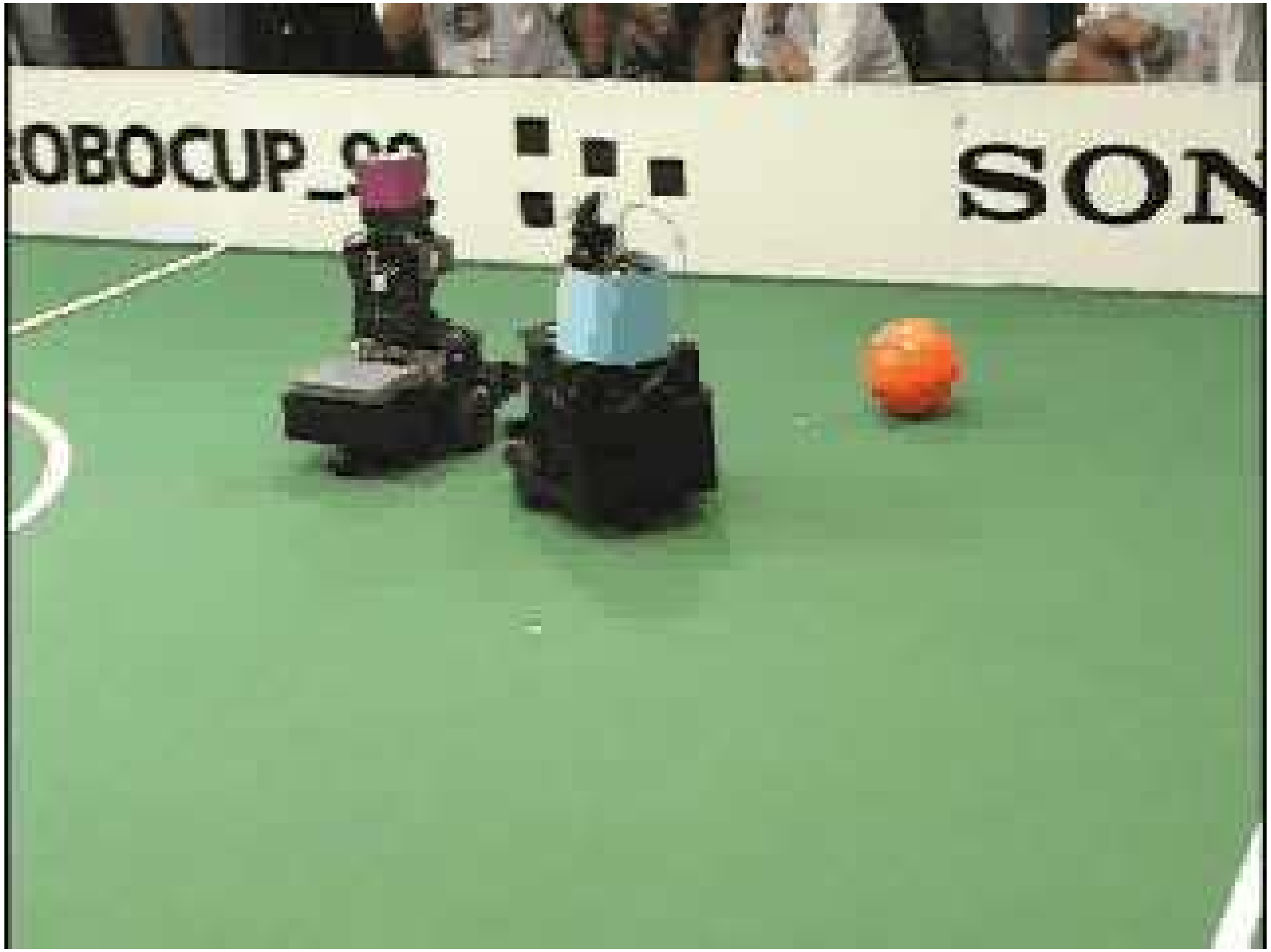
Kalman Filter-based Systems (1)

- [Gutmann et al. 96, 98]:
  - Match LRF scans against map
  - Highly successful in RoboCup mid-size league



Courtesy of S. Gutmann

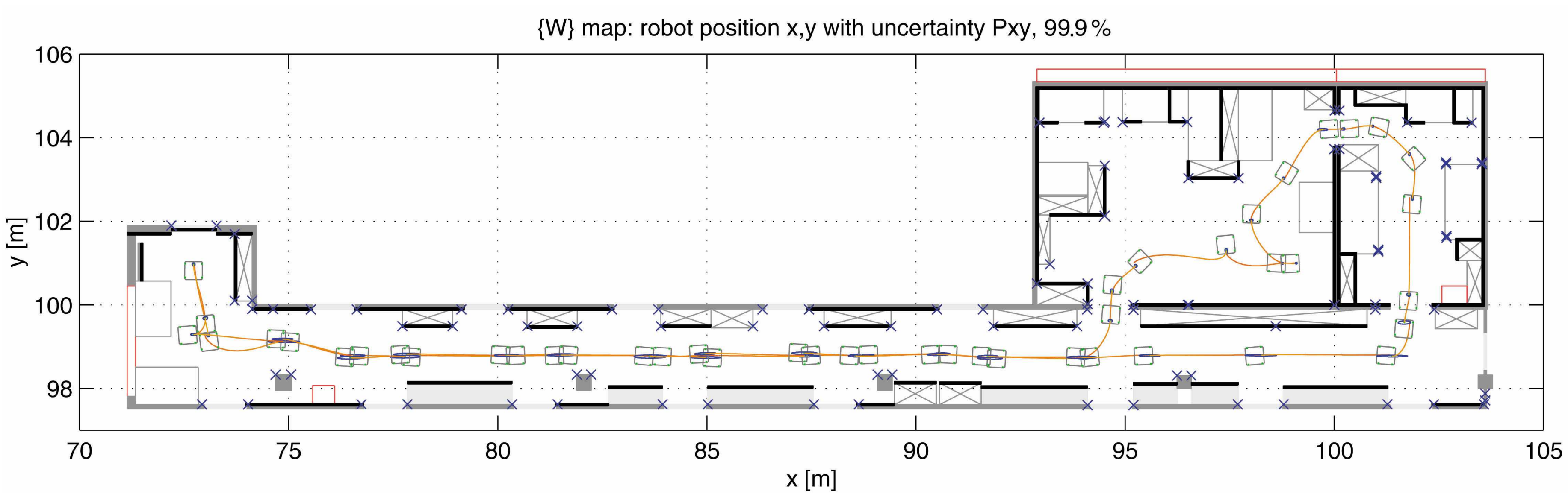
Kalman Filter-based Systems (2)



Courtesy of S. Gutmann

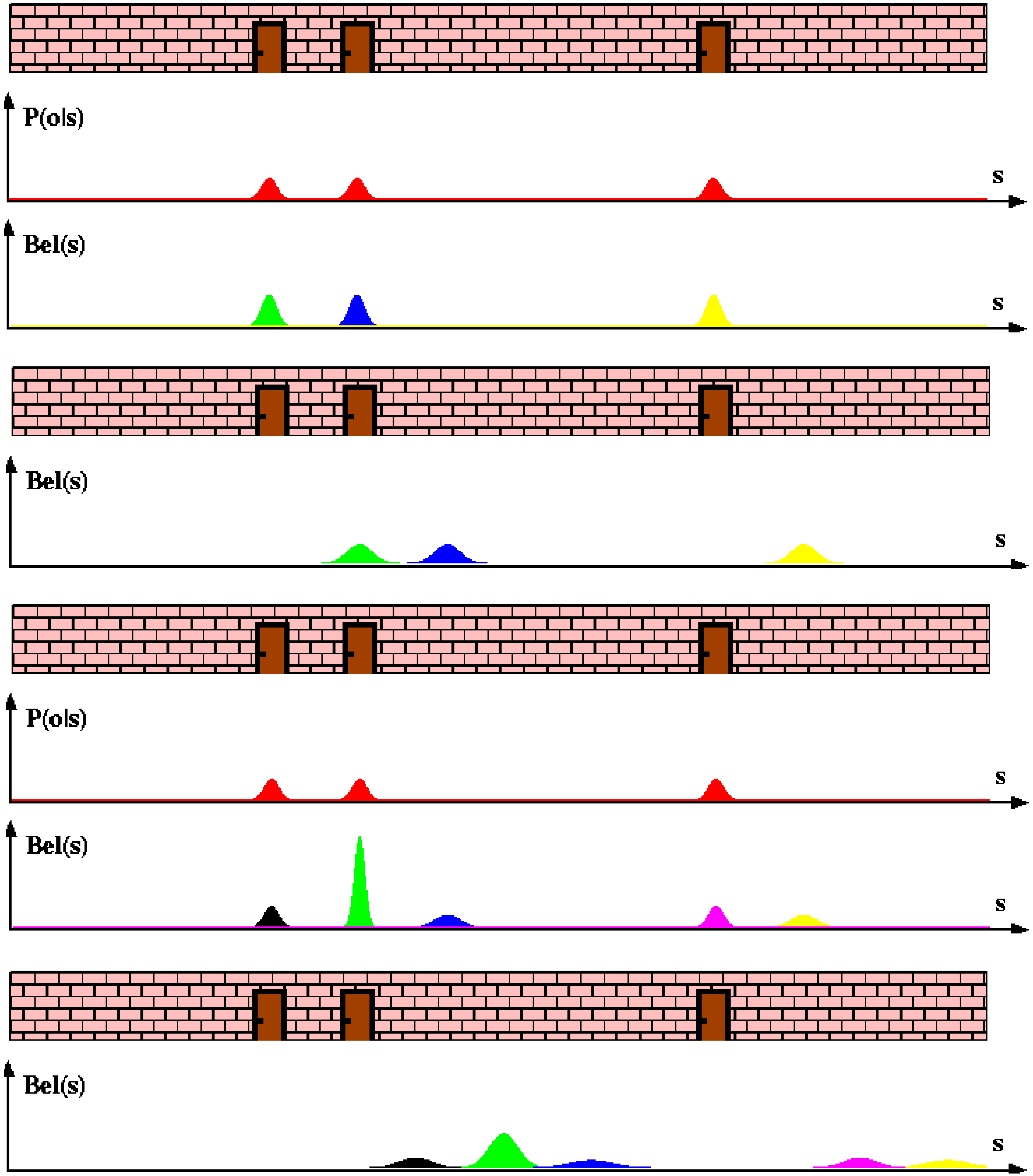
Kalman Filter-based Systems (3)

- [Arras et al. 98]:
  - Laser range-finder and vision
  - High precision (<1cm accuracy)



Courtesy of K. Arras

Multi-hypothesis Tracking



[Cox 92], [Jensfelt, Kristensen 99]

Localization With MHT

- Belief is represented by multiple hypotheses
- Each hypothesis is tracked by a Kalman filter

- **Additional problems**:
  - Data association: Which observation corresponds to which hypothesis?
  - Hypothesis management: When to add / delete hypotheses?
- Huge body of literature on target tracking, motion correspondence etc.

See e.g. [Cox 93]

MHT: Implemented System (1)

- [Jensfelt and Kristensen 99,01]
  - Hypotheses are extracted from LRF scans
  - Each hypothesis has probability of being the correct one:

$$H_i = \{\hat{x}_i, \Sigma_i, P(H_i)\}$$

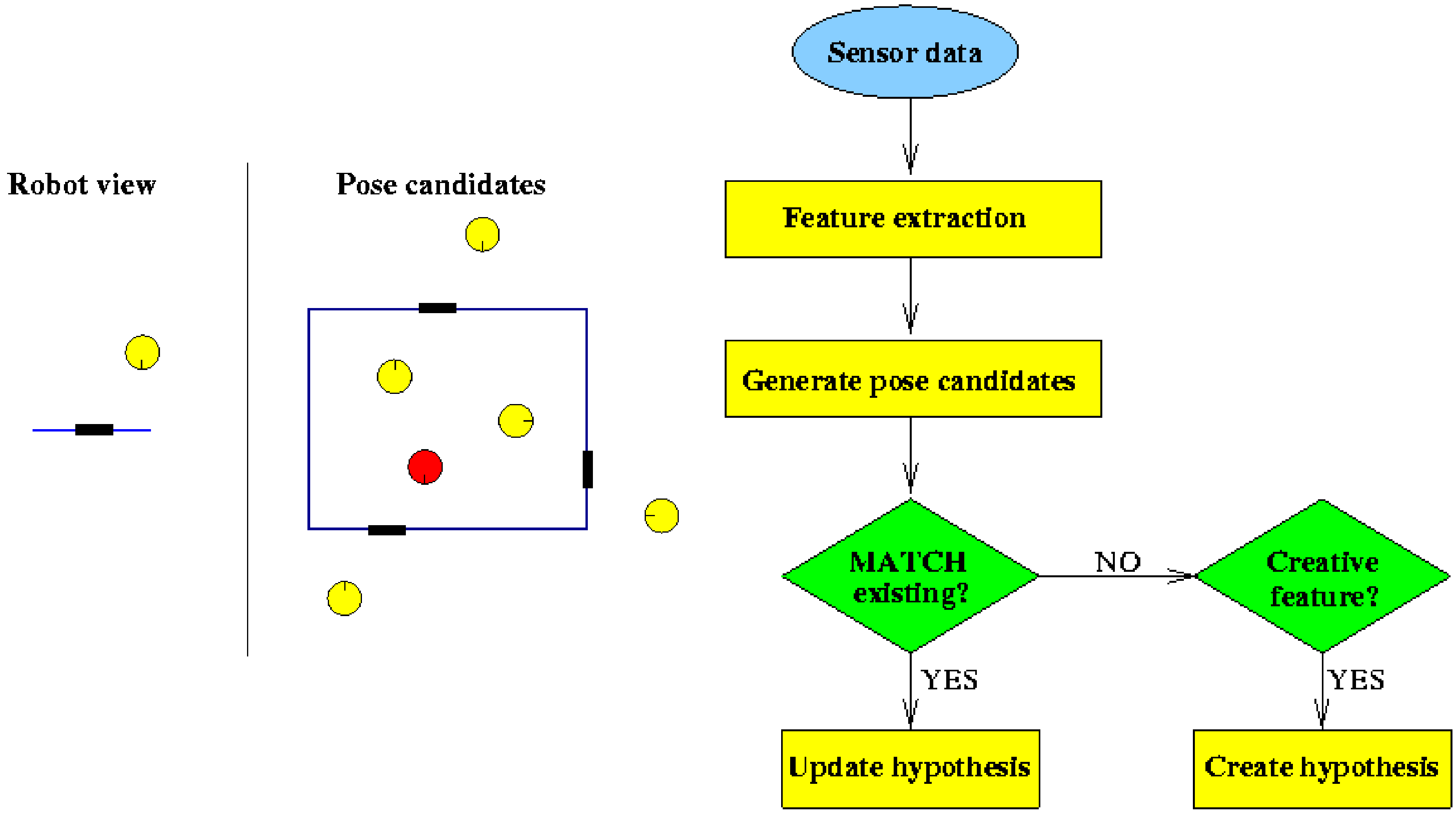
  - Hypothesis probability is computed using Bayes' rule

$$P(H_i \mid s) = \frac{P(s \mid H_i) P(H_i)}{P(s)}$$

  - Hypotheses with low probability are deleted
  - New candidates are extracted from LRF scans

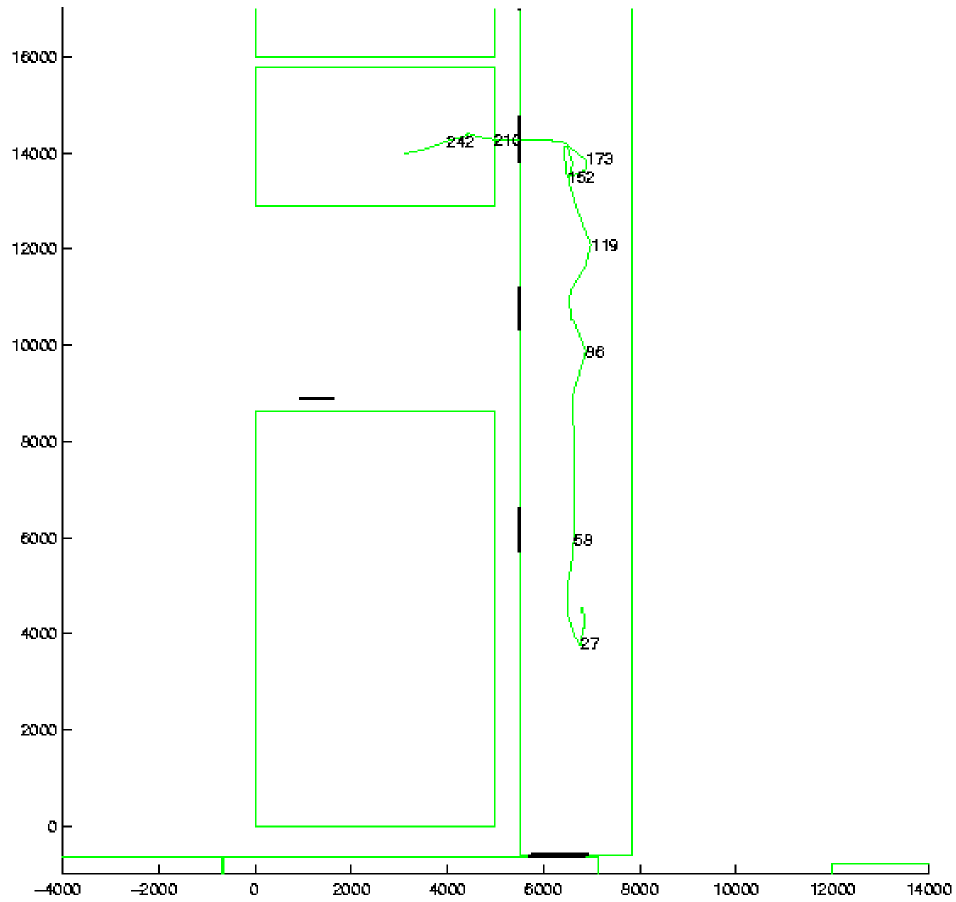
$$C_j = \{z_j, R_j\}$$

MHT: Implemented System (2)

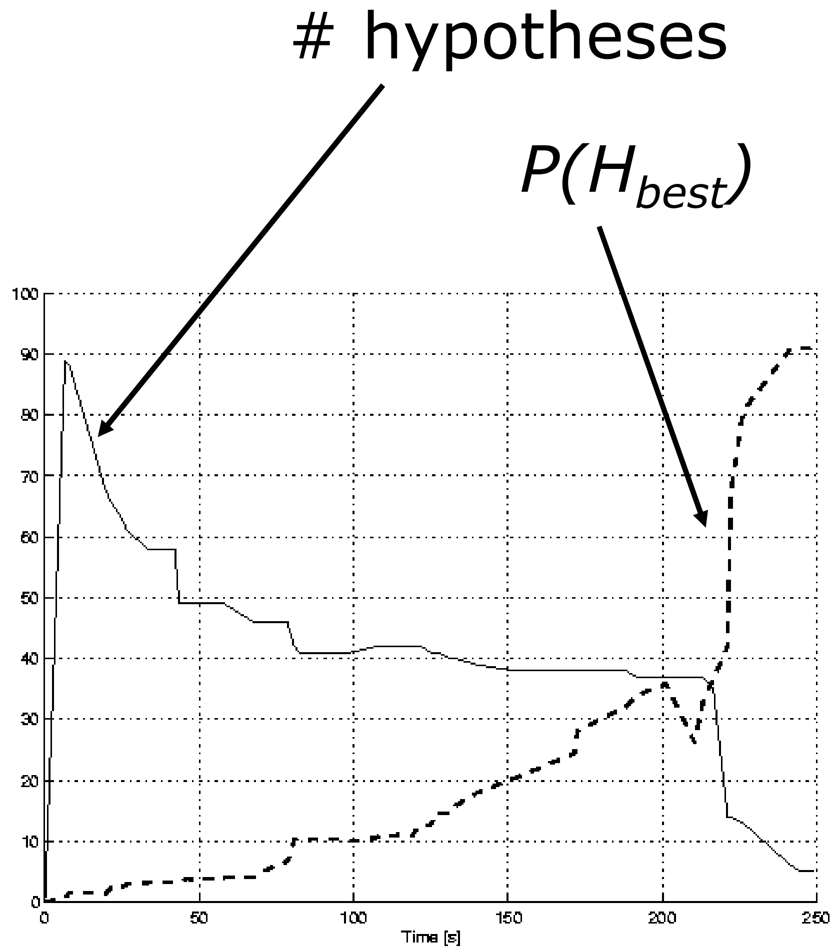


Courtesy of P. Jensfelt and S. Kristensen

MHT: Implemented System (3)
Example run



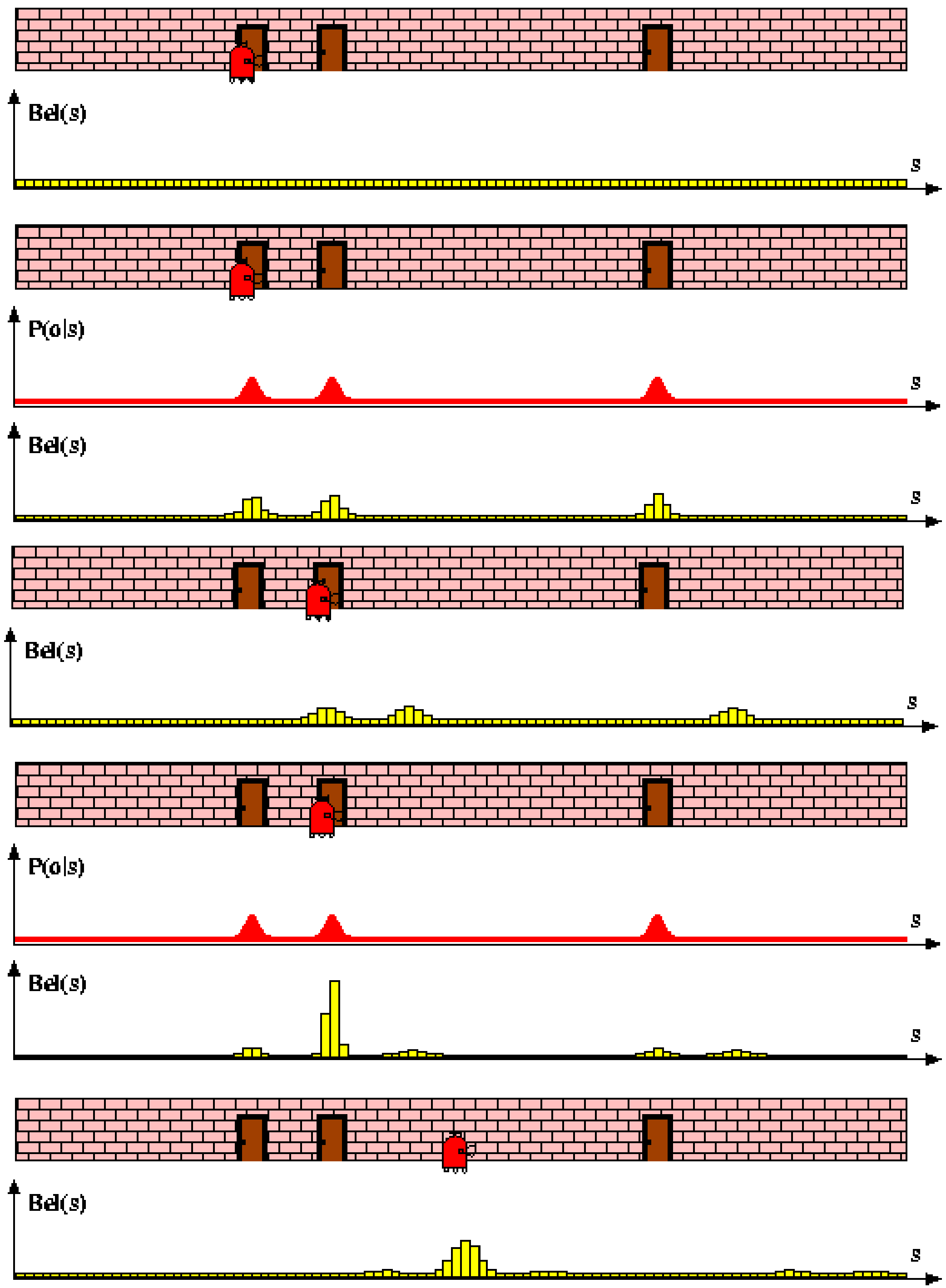
Map and trajectory



Hypotheses vs. time

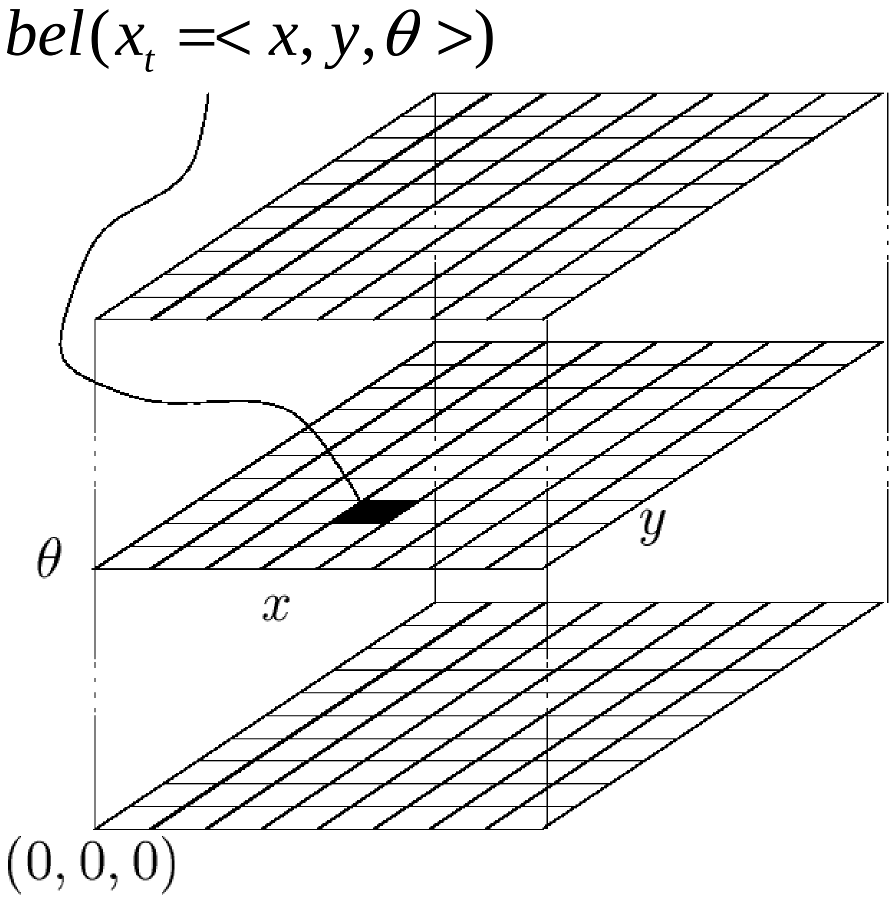
Courtesy of P. Jensfelt and S. Kristensen

Piecewise Constant

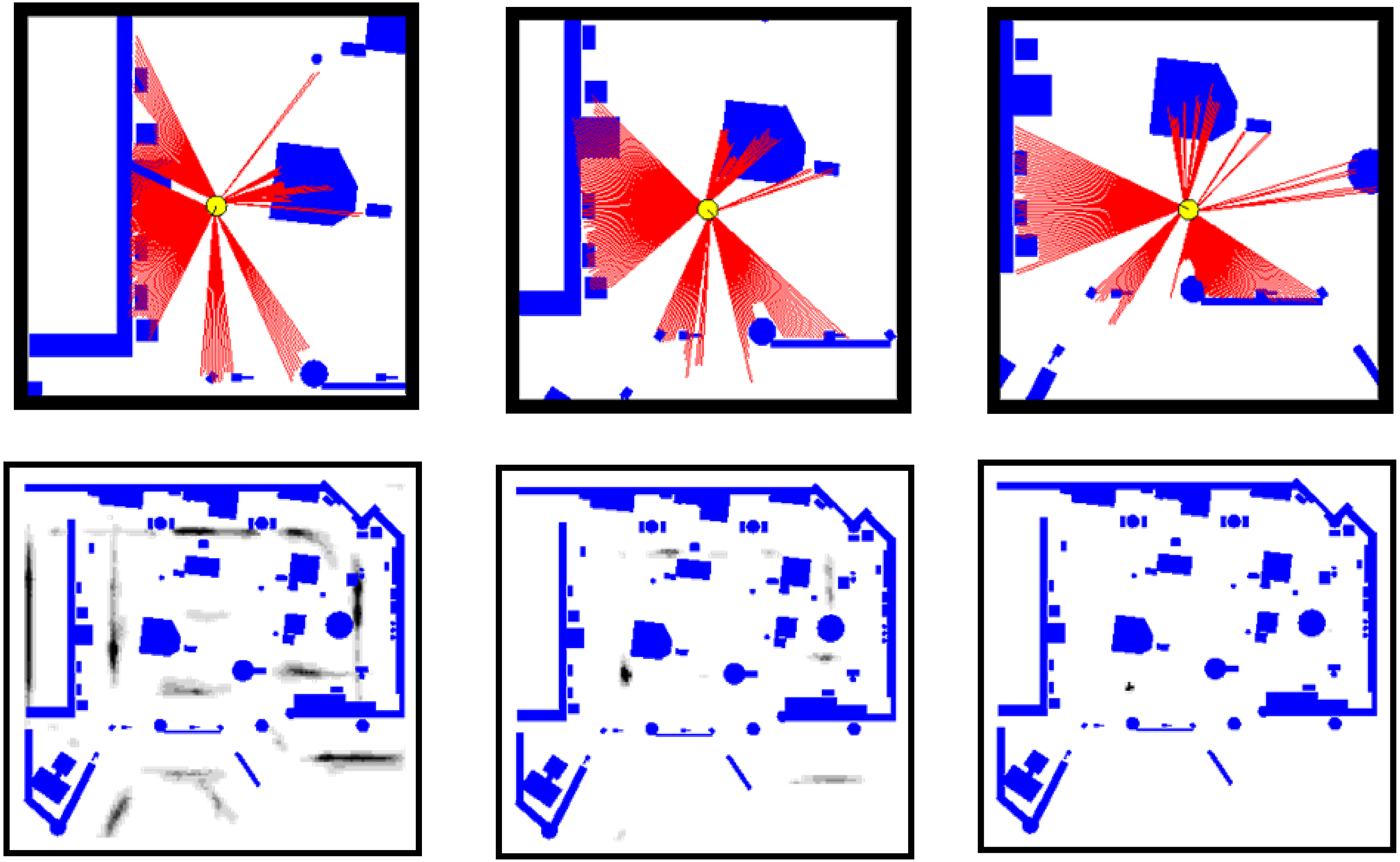


[Burgard et al. 96,98], [Fox et al. 99], [Konolige et al. 99]

Piecewise Constant Representation

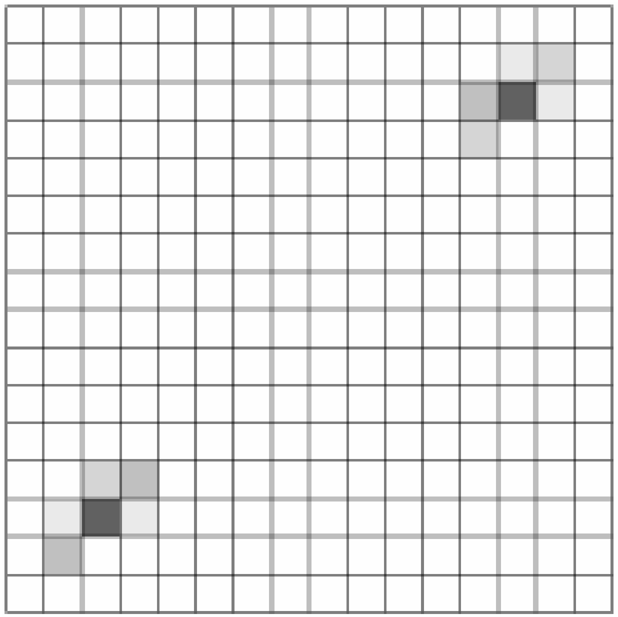


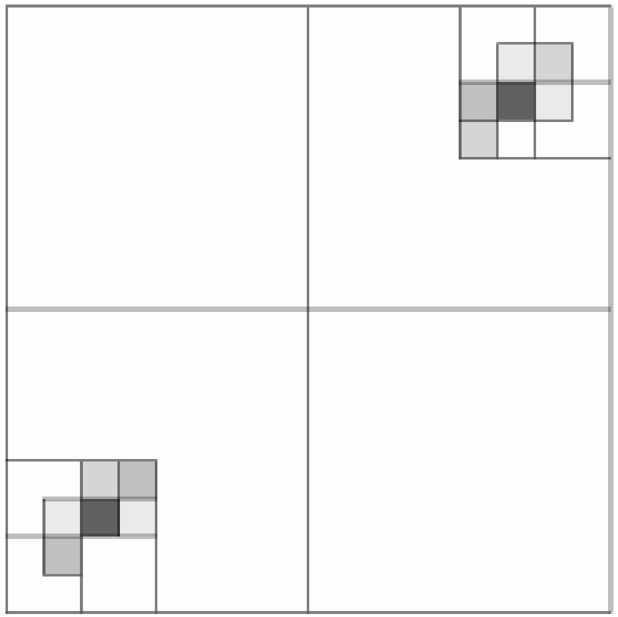
Grid-based Localization



Tree-based Representations (1)

Idea: Represent density using a variant of Octrees





Xavier:
Localization in a Topological Map

[Simmons and Koenig 96]

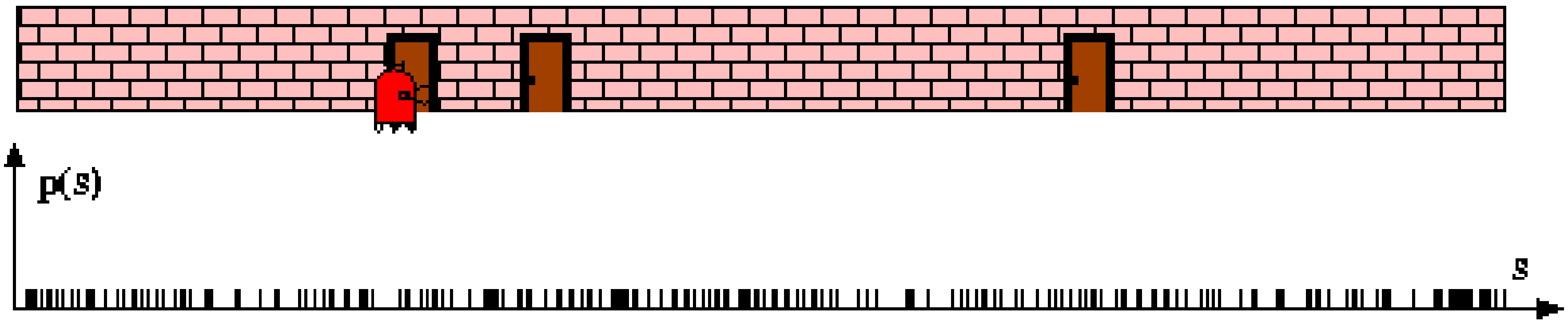
Particle Filters

- Represent density by random samples
- Estimation of non-Gaussian, nonlinear processes

- Monte Carlo filter, Survival of the fittest, Condensation, Bootstrap filter, Particle filter

- Filtering: [Rubin, 88], [Gordon et al., 93], [Kitagawa 96]
- Computer vision: [Isard and Blake 96, 98]
- Dynamic Bayesian Networks: [Kanazawa et al., 95]

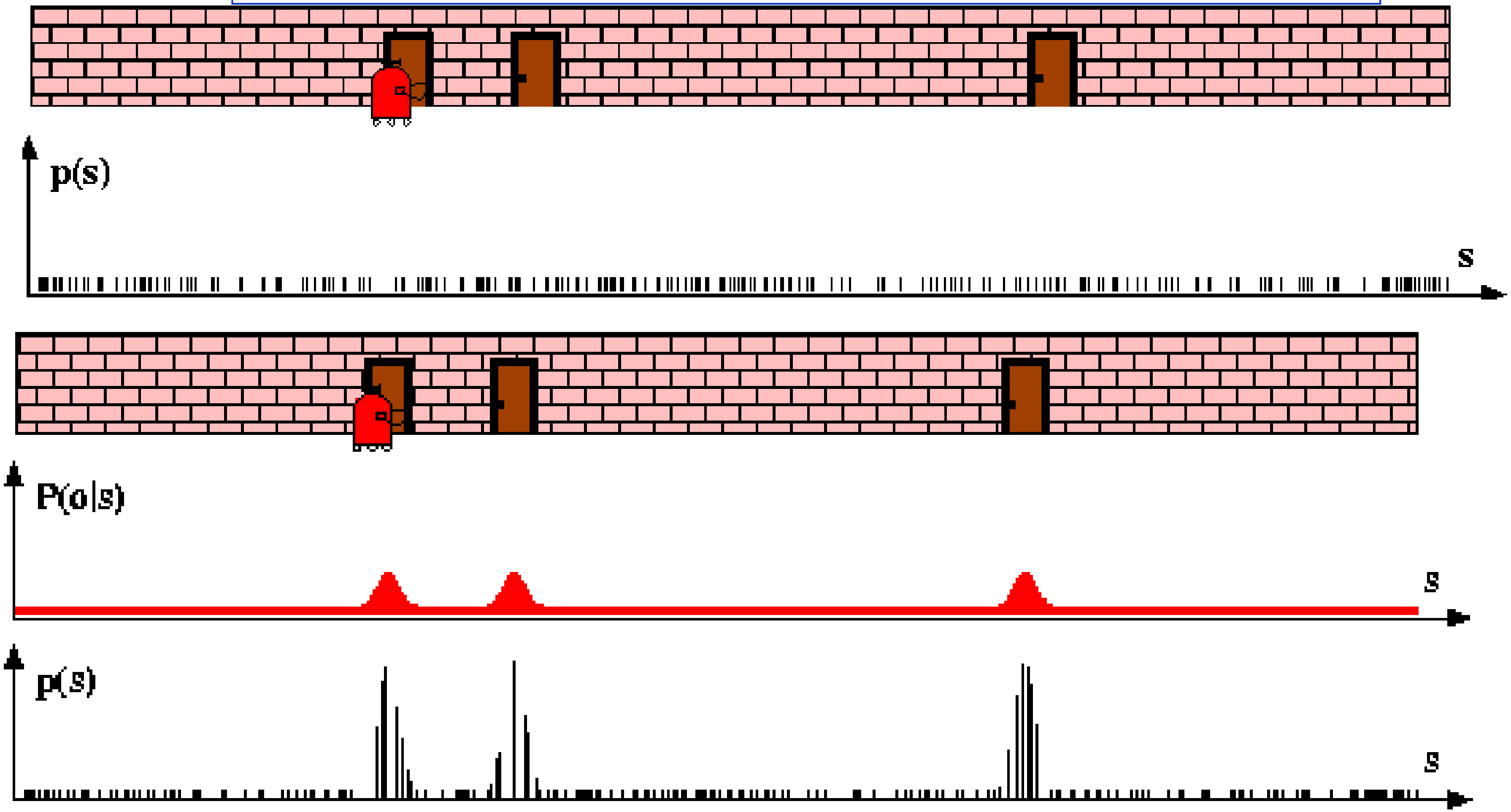
MCL: Global Localization



MCL: Sensor Update

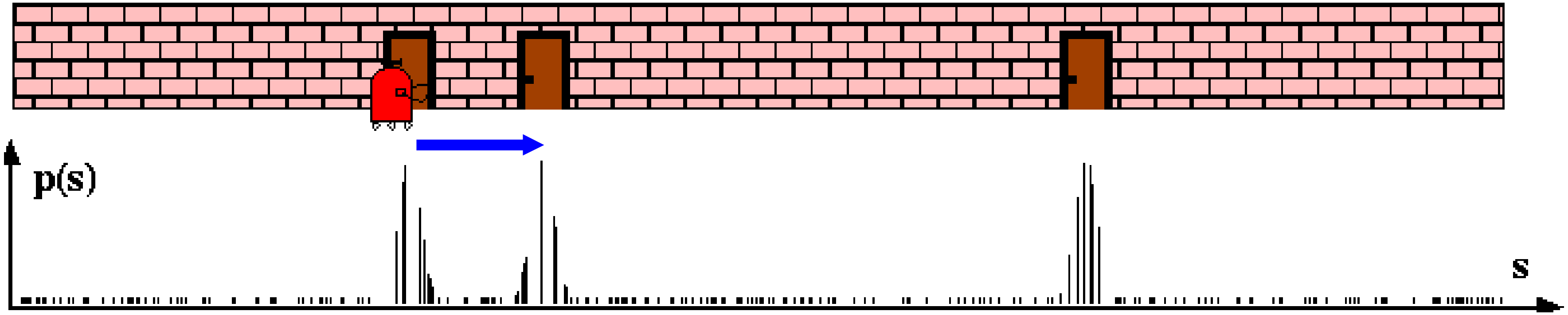
$$Bel(x) \leftarrow \alpha\, p(z \mid x)\, Bel^-(x)$$

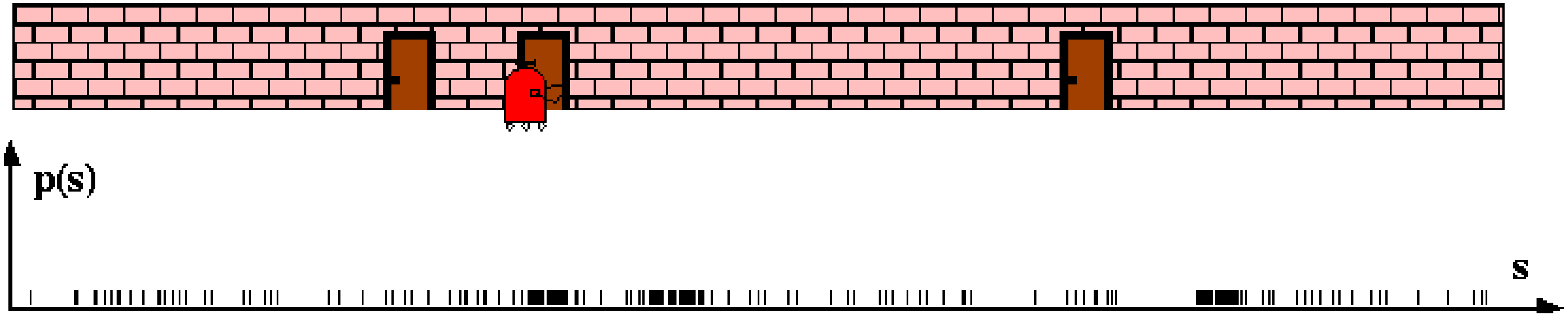
$$w \leftarrow \frac{\alpha\, p(z \mid x)\, Bel^-(x)}{Bel^-(x)} = \alpha\, p(z \mid x)$$



MCL: Robot Motion

$$Bel^{-}(x) \leftarrow \int p(x \mid u, x')\, Bel(x')\, \mathrm{d}x'$$





MCL: Sensor Update

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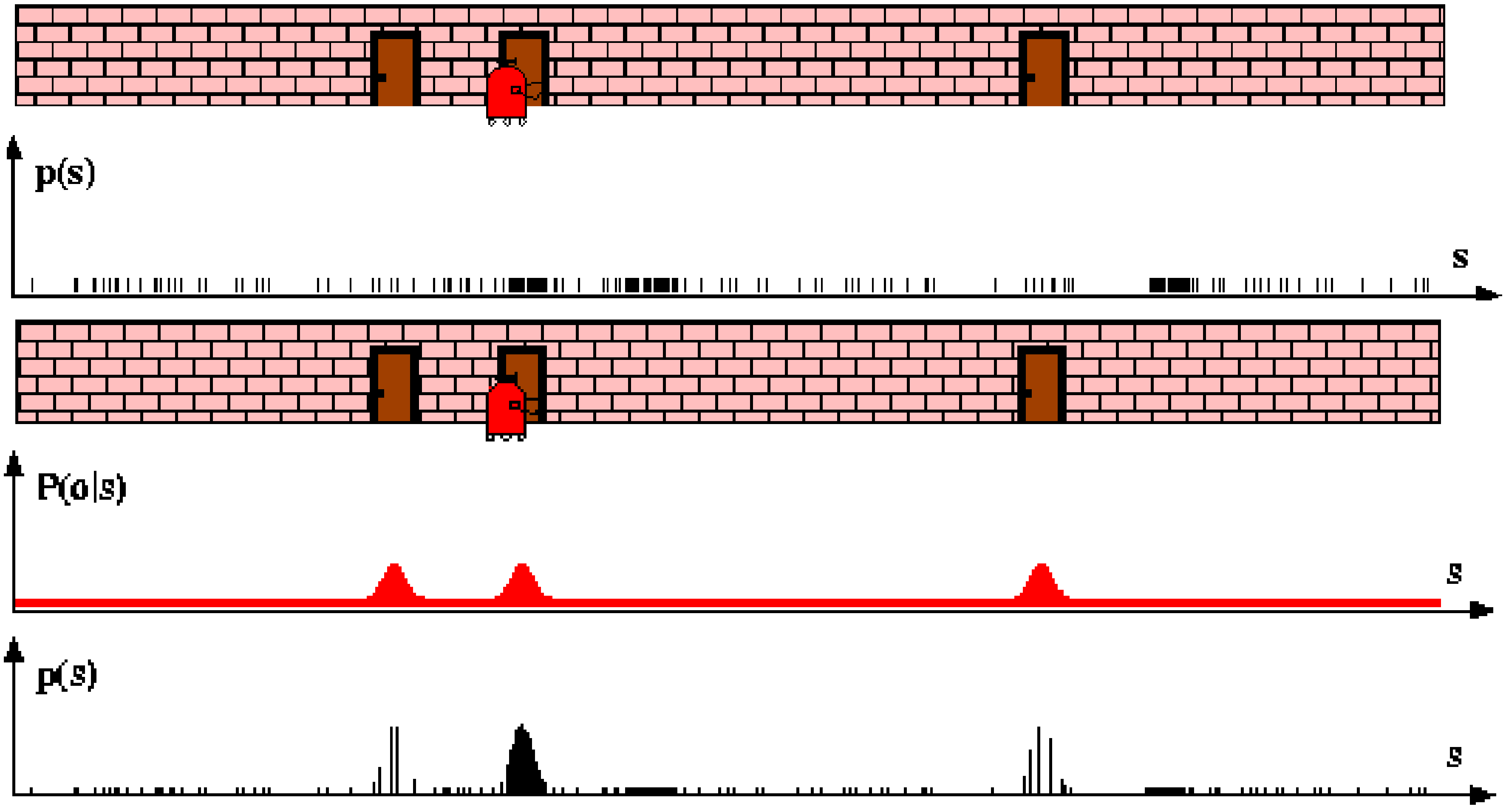
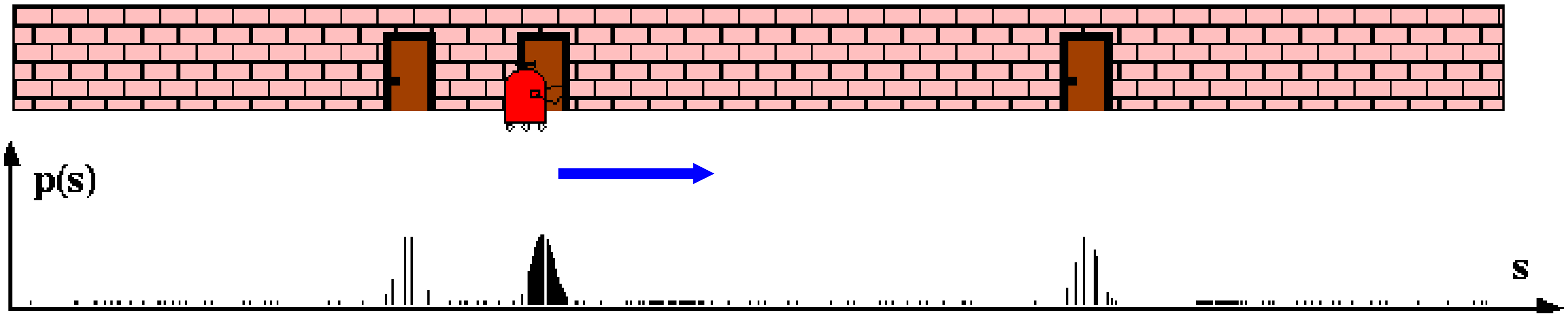
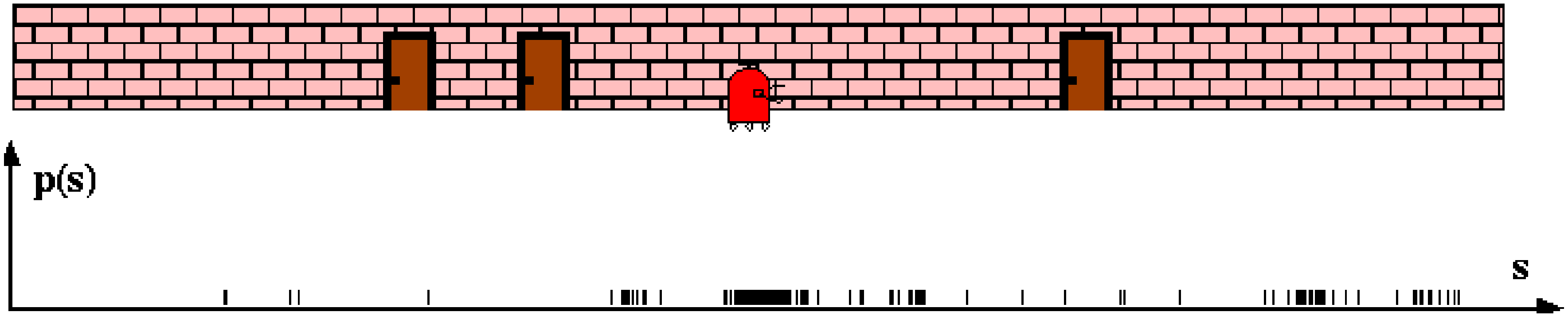


MCL: Robot Motion

$$Bel^{-}(x) \leftarrow \int p(x \mid u, x')\, Bel(x')\ \mathrm{d}x'$$





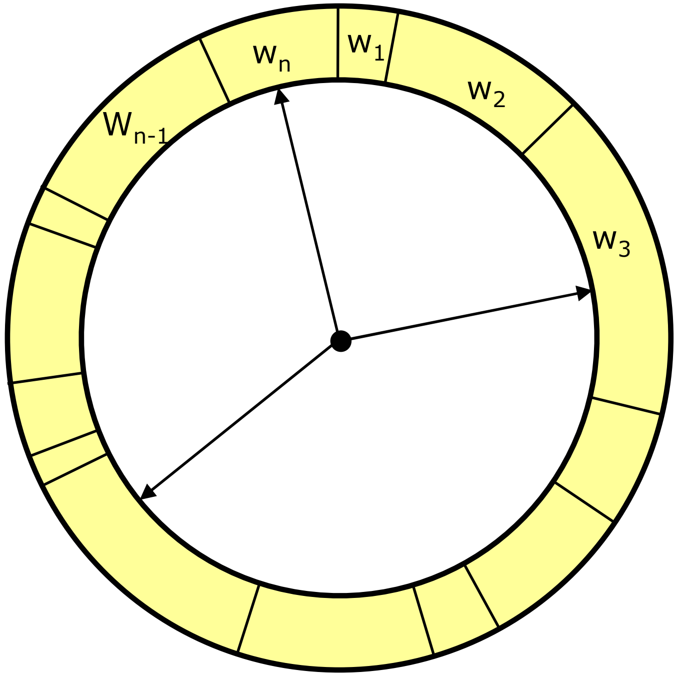
Particle Filter Algorithm

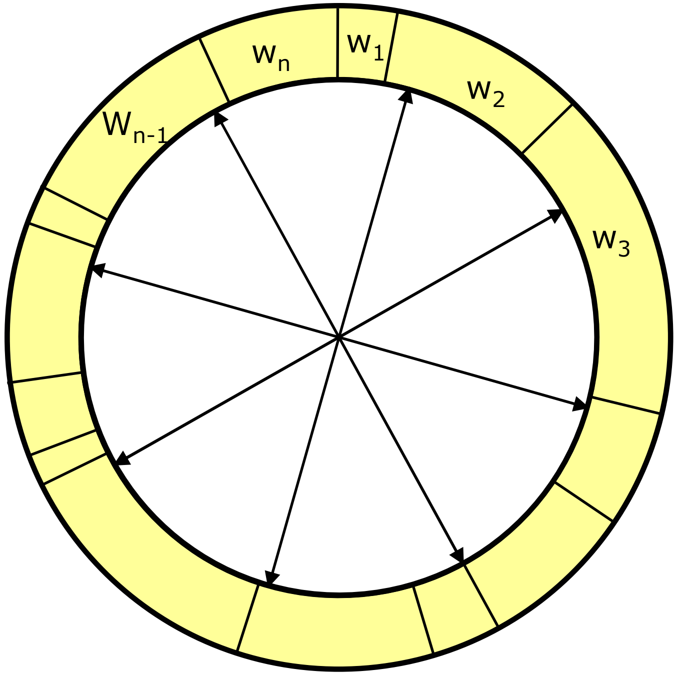
1. Algorithm **particle_filter**(S_{t-1}, u_{t-1} z_t):
2. $S_t = \emptyset, \quad \eta = 0$
3. **For** $i = 1 \dots n$ — ***Generate new samples***
4. Sample index *j(i)* from the discrete distribution given by w_{t-1}
5. Sample x_t^i from $p(x_t \mid x_{t-1}, u_{t-1})$ using $x_{t-1}^{j(i)}$ and u_{t-1}
6. $w_t^i = p(z_t \mid x_t^i)$ — ***Compute importance weight***
7. $\eta = \eta + w_t^i$ — ***Update normalization factor***
8. $S_t = S_t \cup \{< x_t^i, w_t^i >\}$ — ***Insert***
9. **For** $i = 1 \dots n$
10. $w_t^i = w_t^i / \eta$ — ***Normalize weights***

Resampling

- **Given**: Set S of weighted samples.
- **Wanted** : Random sample, where the probability of drawing x_i is given by w_i.
- Typically done n times with replacement to generate new sample set S'.

Resampling



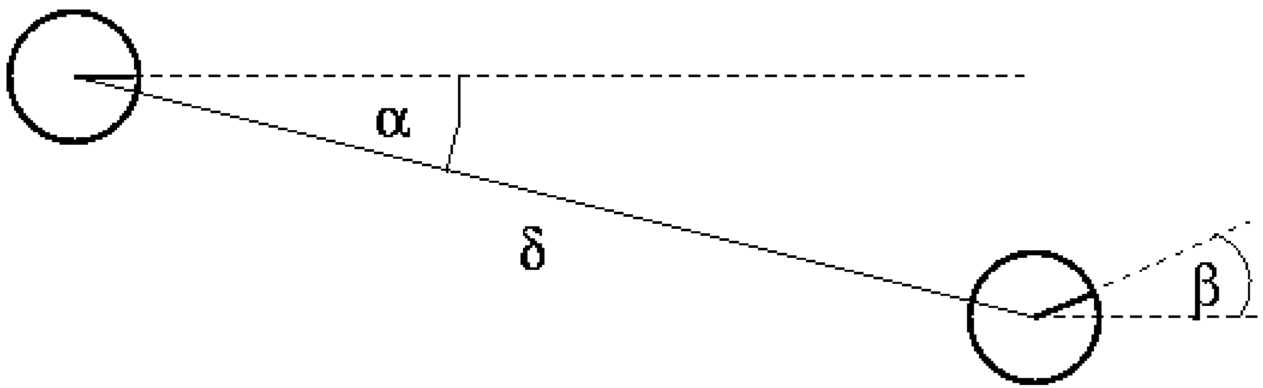


- Roulette wheel
- Binary search, log n

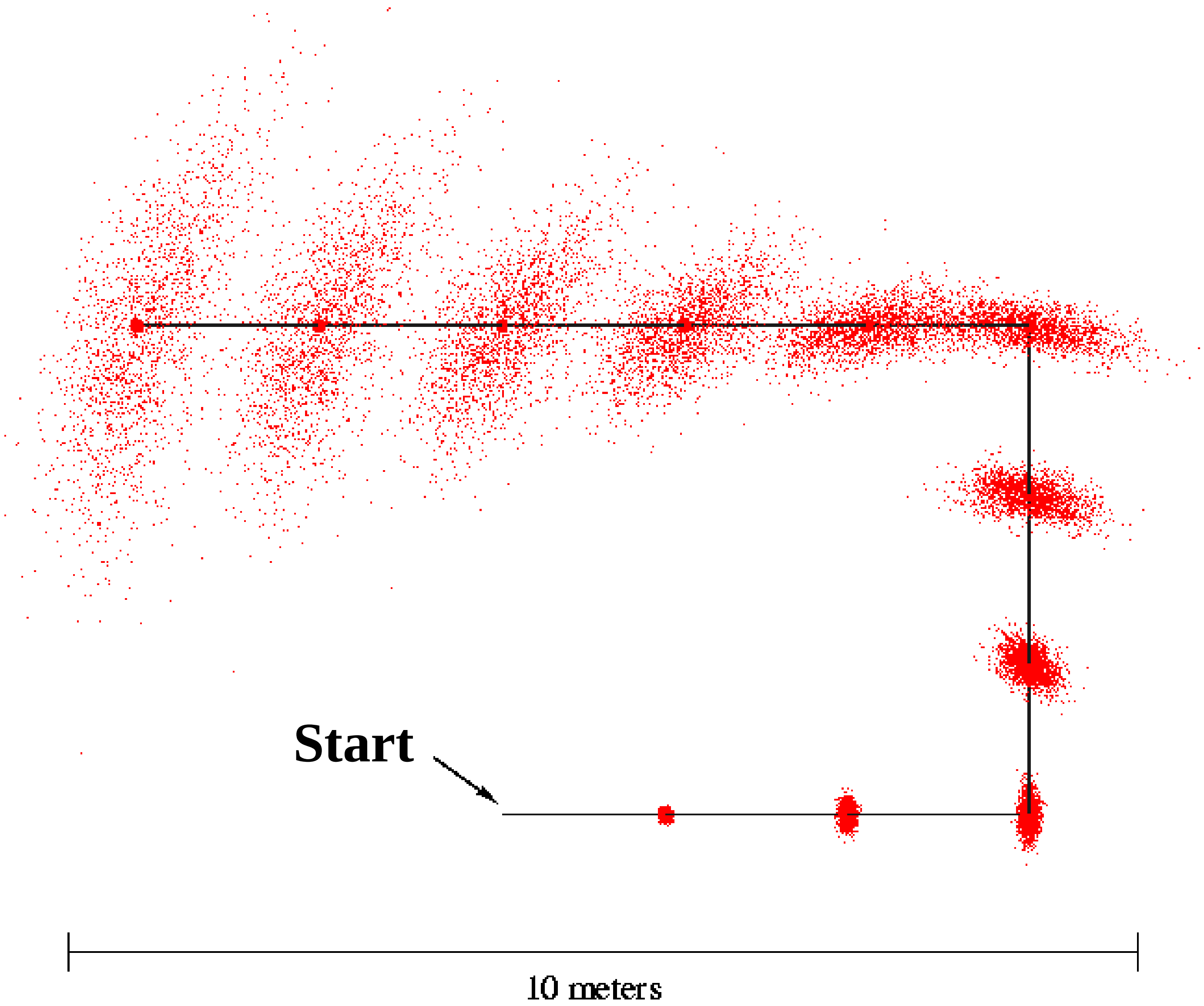
- Stochastic universal sampling
- Systematic resampling
- Linear time complexity
- Easy to implement, low variance

Motion Model $p(x_t \mid a_{t-1}, x_{t-1})$

Model odometry error as Gaussian noise on α, β, and δ



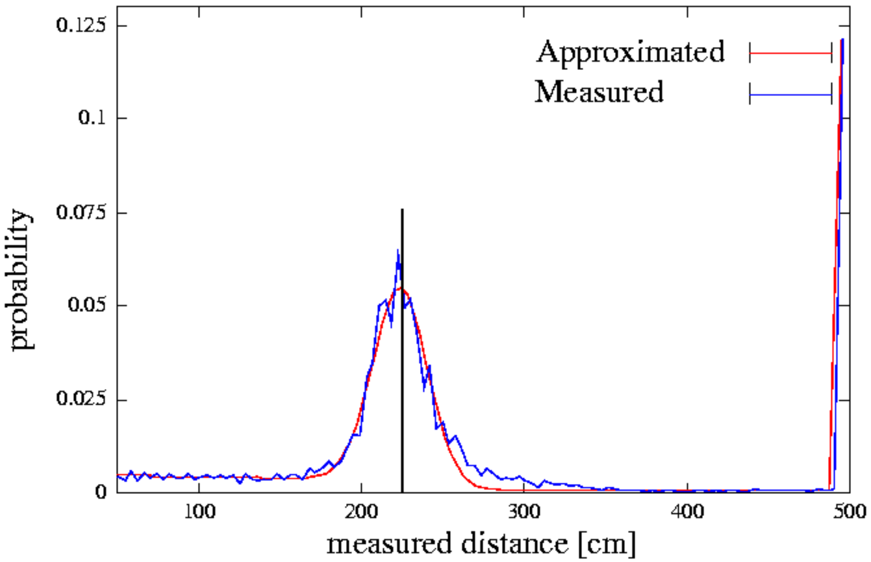
Motion Model $p(x_t \mid a_{t-1}, x_{t-1})$



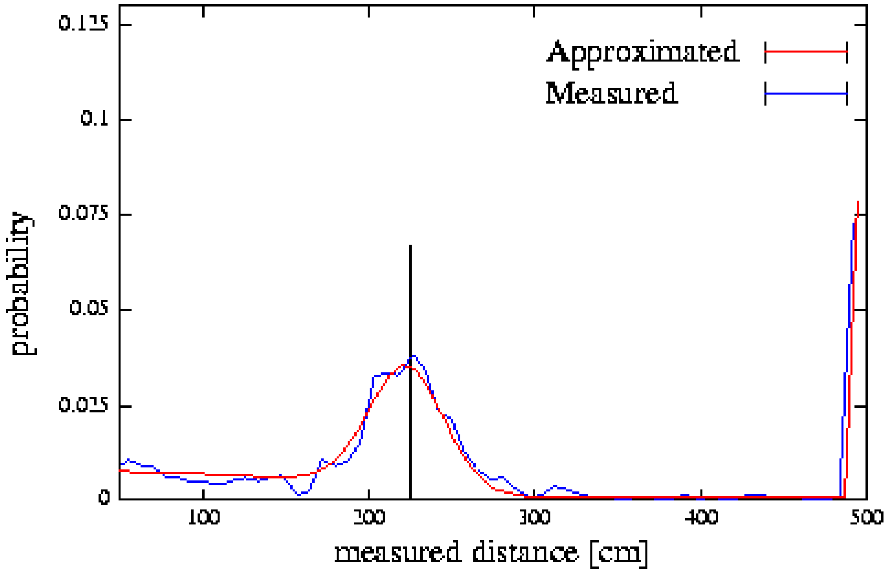
Model for Proximity Sensors

The sensor is reflected either by a known or by an unknown obstacle:

$$P(d_i \mid l) = 1 - (1 - (1 - \sum_{j<i} P_u(d_j))\, c_d\, P_m(d_i \mid l)) \cdot (1 - (1 - \sum_{j<i} P(d_j))\, c_r)$$

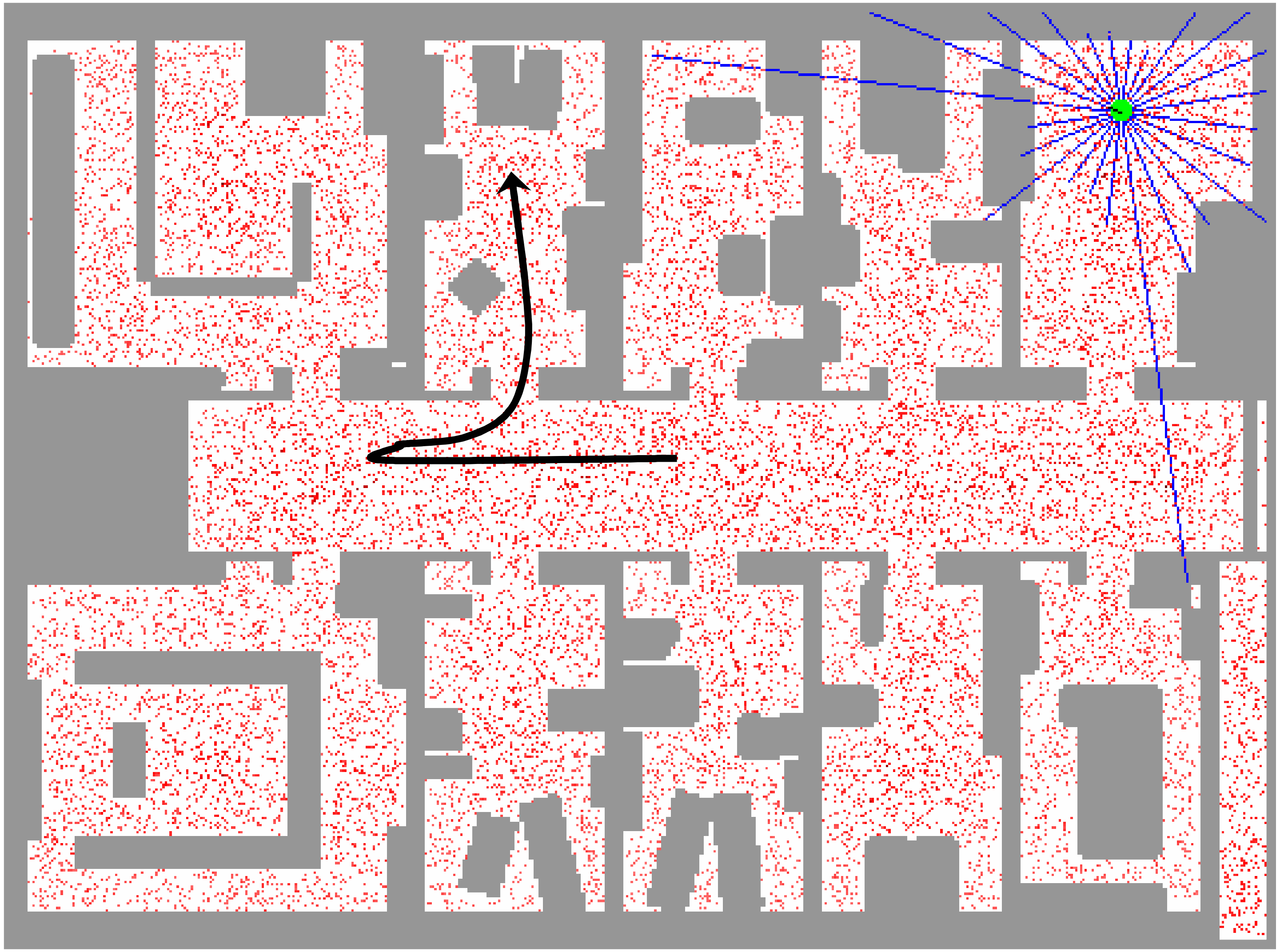


Laser sensor



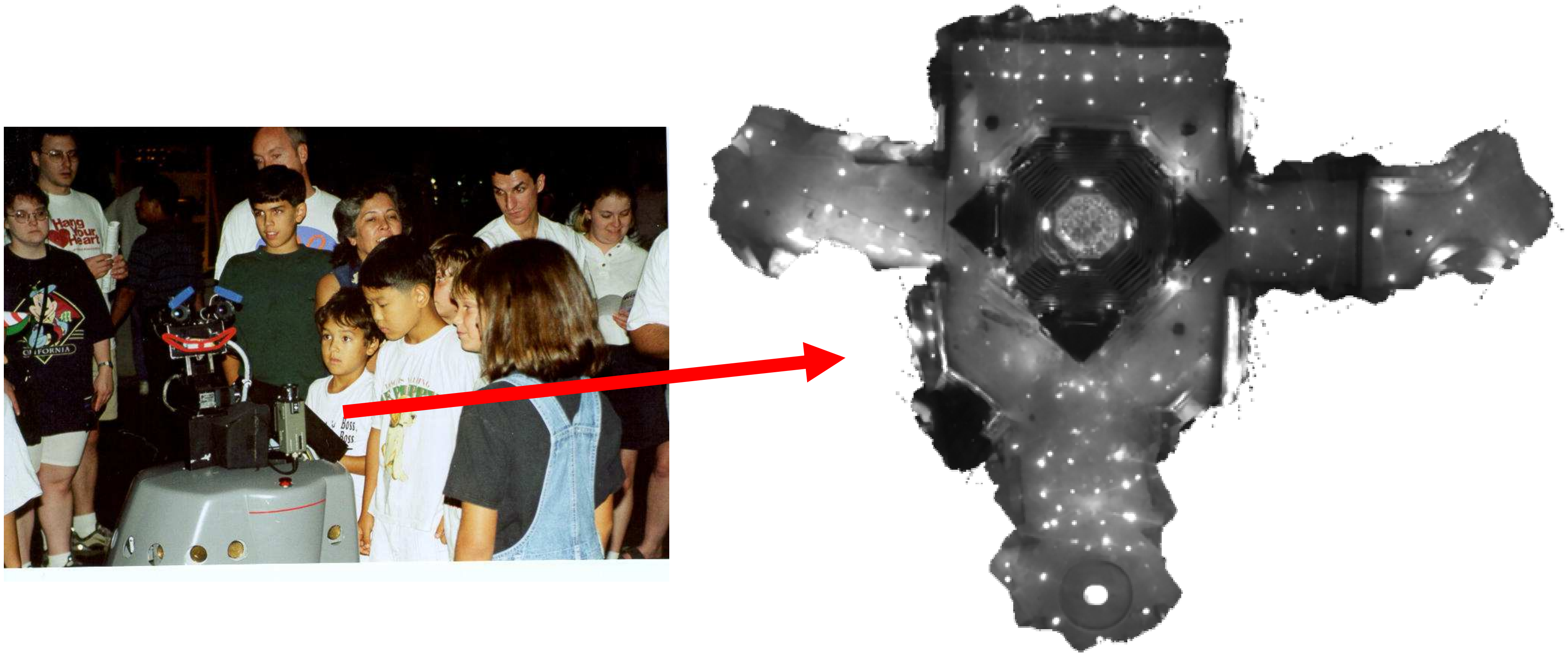
Sonar sensor

MCL: Global Localization (Sonar)



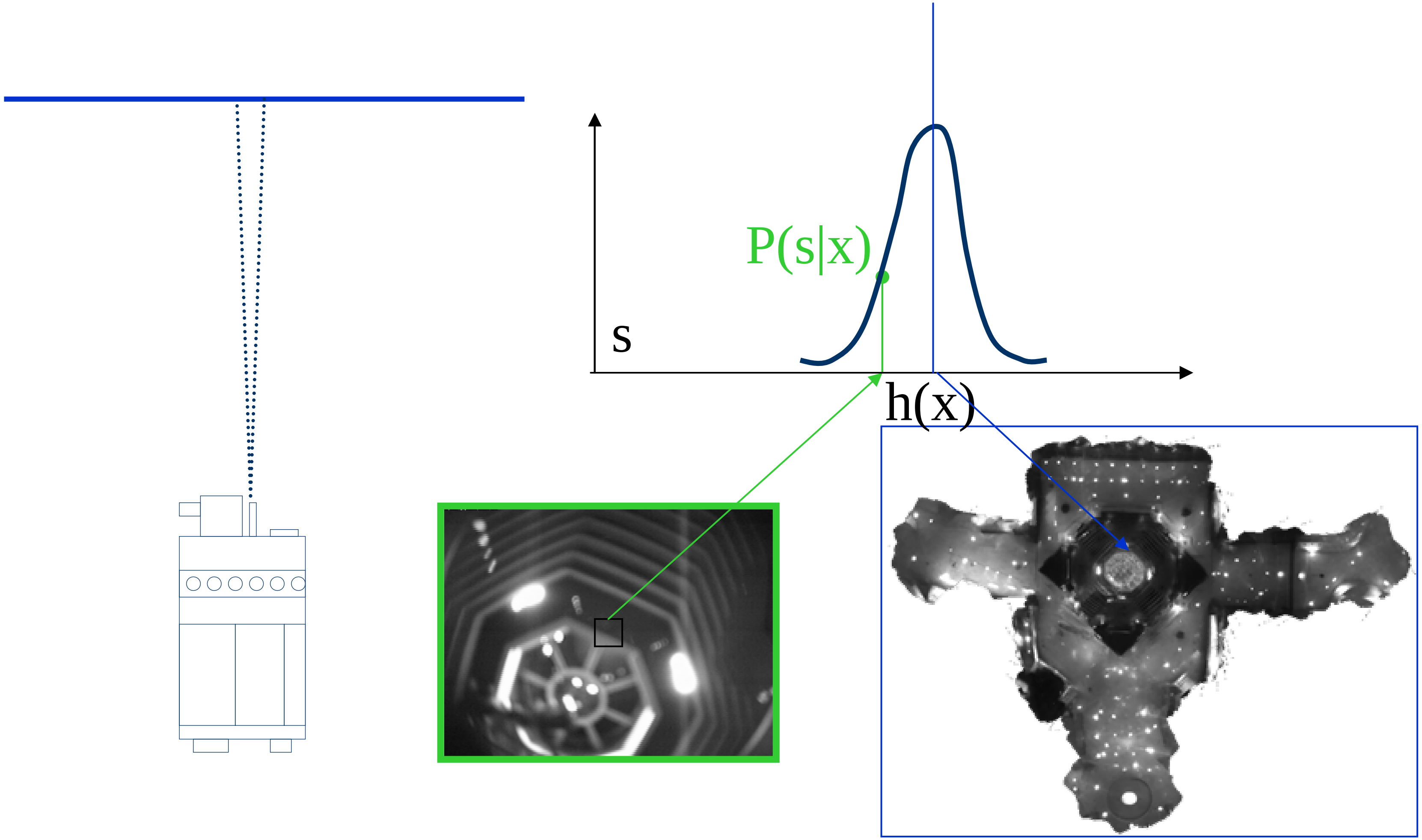
[Fox et al., 99]

Using Ceiling Maps for Localization

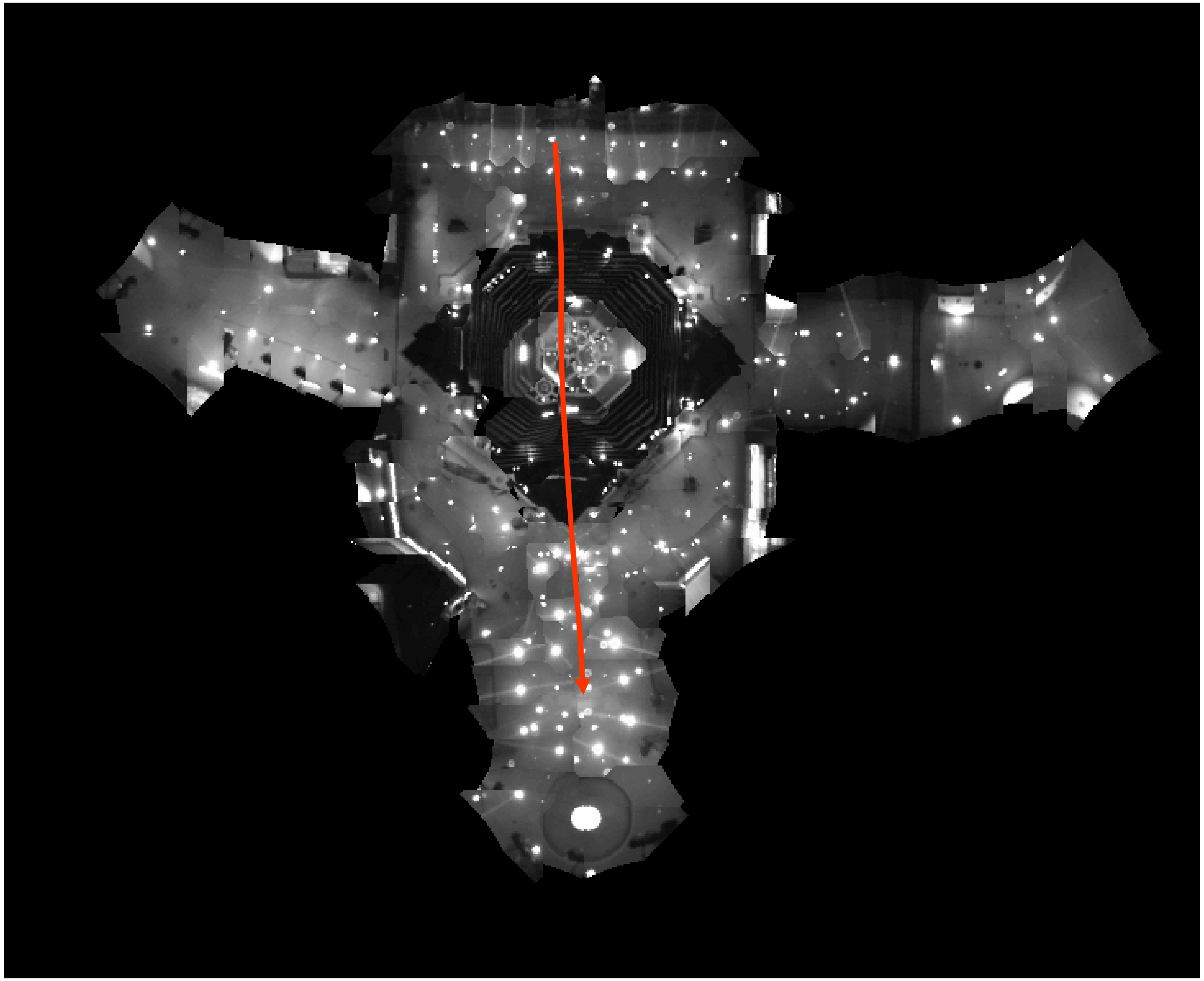


[Dellaert et al. 99]

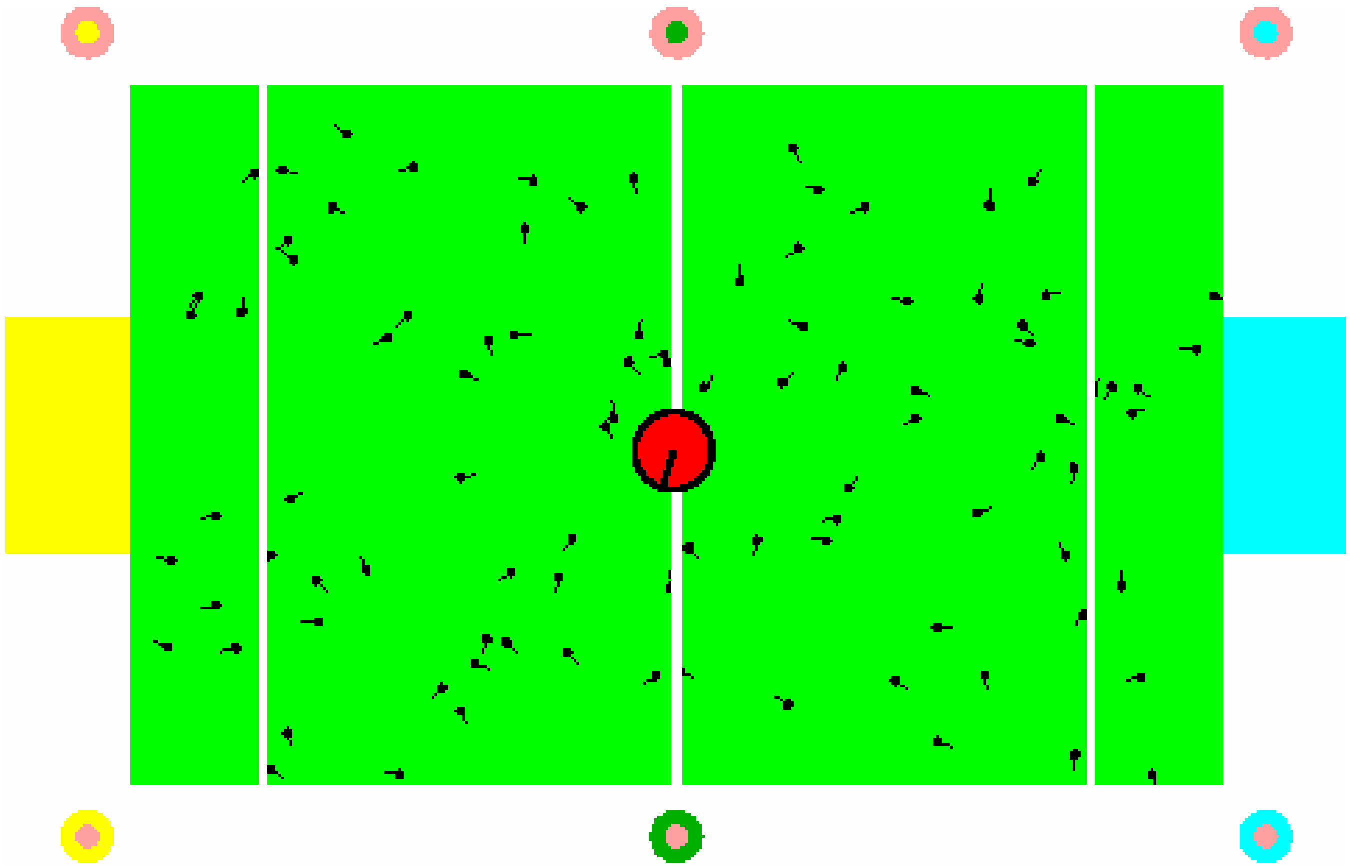
Vision-based Localization



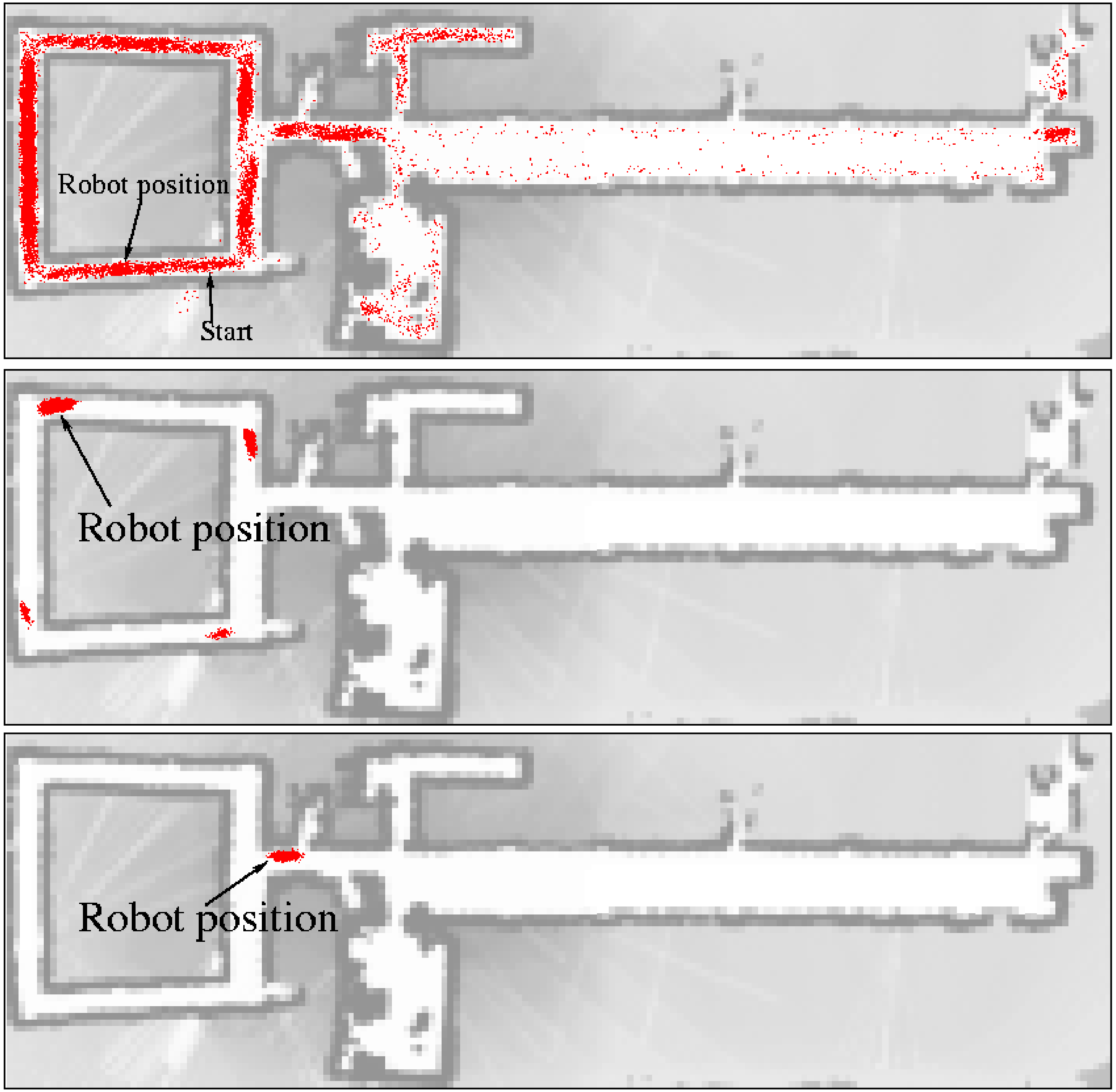
MCL: Global Localization Using Vision



Localization for AIBO robots



Adaptive Sampling



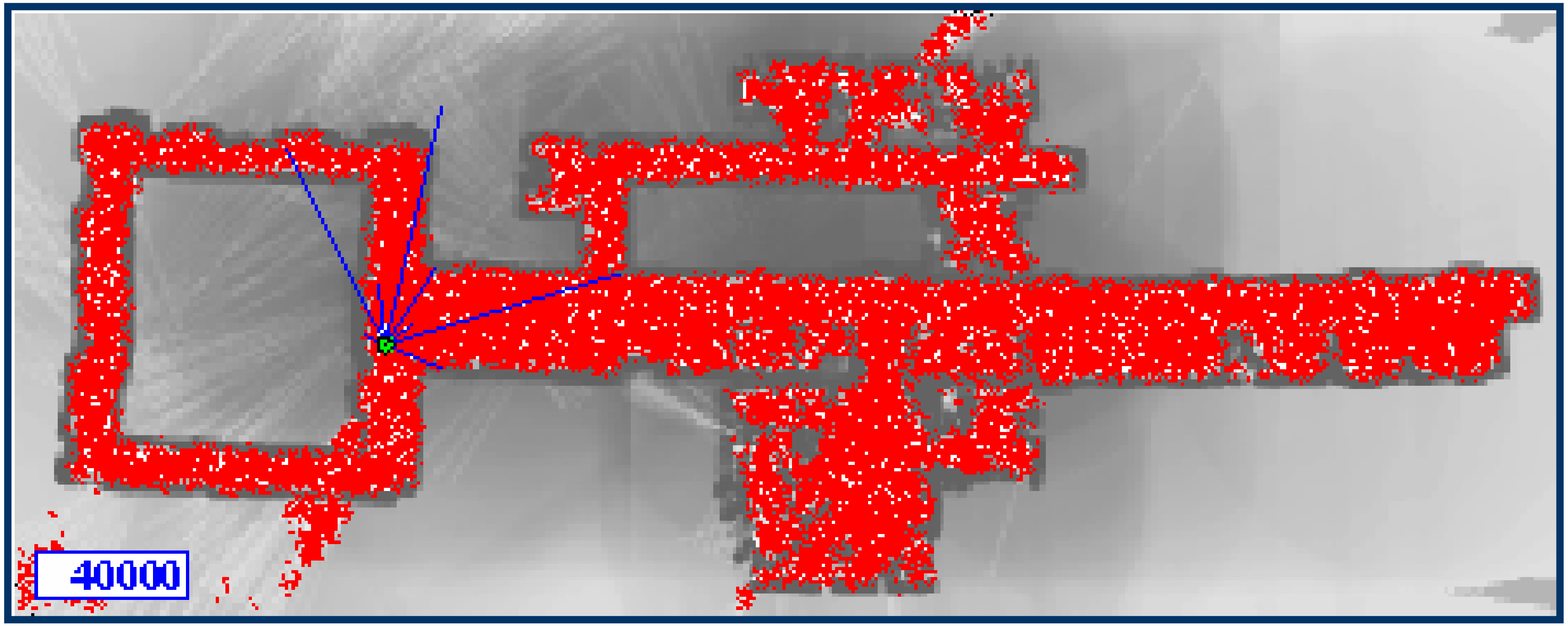
KLD-sampling

- **Idea**:
  - Assume we know the true belief.
  - Represent this belief as a multinomial distribution.
  - Determine number of samples such that we can guarantee that, with probability (1- δ), the KL-distance between the true posterior and the sample-based approximation is less than ε.

- **Observation**:
  - For fixed δ and ε, number of samples only depends on number k of bins with support:

$$n = \frac{1}{2\varepsilon} \mathrm{X}^2(k-1, 1-\delta) \cong \frac{k-1}{2\varepsilon} \left\{ 1 - \frac{2}{9(k-1)} + \sqrt{\frac{2}{9(k-1)}} z_{1-\delta} \right\}^3$$

MCL: Adaptive Sampling (Sonar)



Particle Filters for Robot Localization (Summary)

- Approximate Bayes Estimation/Filtering
  - Full posterior estimation
  - Converges in O(1/√#samples) [Tanner'93]
  - Robust: multiple hypotheses with degree of belief
  - Efficient in low-dimensional spaces: focuses computation where needed
  - Any-time: by varying number of samples
  - Easy to implement

Localization Algorithms - Comparison

	Kalman filter	Multi-hypothesis tracking	Topological maps	Grid-based (fixed/variable)	Particle filter
Sensors	Gaussian	Gaussian	Features	Non-Gaussian	Non-Gaussian
Posterior	Gaussian	Multi-modal	Piecewise constant	Piecewise constant	Samples
Efficiency (memory)	++	++	++	-/+	+/++
Efficiency (time)	++	++	++	o/+	+/++
Implementation	+	o	+	+/o	++
Accuracy	++	++	-	+/++	++
Robustness	-	+	+	++	+/++
Global localization	No	Yes	Yes	Yes	Yes

Localization: Lessons Learned

- Probabilistic Localization = Bayes filters
- Particle filters: Approximate posterior by random samples
- Extensions:
  - Filter for dynamic environments
  - Safe avoidance of invisible hazards
  - People tracking
  - Recovery from total failures
  - Active Localization
  - Multi-robot localization