Motivation

- So far: All players move simultaneously, and then the outcome is determined.
- Often in practice: Several moves in sequence (e.g. in chess).
- Extensive games (with perfect information) reflect such situations by modeling games as game trees.
- Idea: Players have several decision points where they can decide how to play.
- Strategies: Mappings from decision points in the game tree to actions to be played.
**Definition (Extensive game with perfect information)**

An extensive game with perfect information is a tuple $\Gamma = (\mathcal{N}, H, P, (u_i)_{i \in \mathcal{N}})$ that consists of:

- A finite non-empty set $\mathcal{N}$ of players.
- A set $H$ of (finite or infinite) sequences, called histories, such that:
  - it contains the empty sequence $\langle \rangle \in H$.
  - $H$ is closed under prefixes: if $\langle a^1, \ldots, a^k \rangle \in H$ for some $k \in \mathbb{N} \cup \{\infty\}$, and $l < k$, then also $\langle a^1, \ldots, a^l \rangle \in H$, and
  - $H$ is closed under limits: if for some infinite sequence $\langle a^i \rangle_{i=1}^{\infty}$, we have $\langle a^k \rangle_{k=1}^{\infty} \in H$ for all $k \in \mathbb{N}$, then $\langle a^k \rangle_{k=1}^{\infty} \in H$.

All infinite histories and all histories $\langle a^k \rangle_{k=1}^{\infty} \in H$, for which there is no $a^{k+1}$ such that $\langle a^k \rangle_{k=1}^{\infty} \in H$ are called terminal histories $Z$. Components of a history are called actions.

---

**Example (Division game)**

- Two identical objects should be divided among two players.
- Player 1 proposes an allocation.
- Player 2 agrees or rejects.
  - On agreement: Allocation as proposed.
  - On rejection: Nobody gets anything.

![Division Game Diagram](image)

- $N = \{1, 2\}$
- $H = \{\langle \rangle, \langle (2,0) \rangle, \langle (1,1) \rangle, \langle (0,2) \rangle, \langle (2,0), y \rangle, \langle (2,0), n \rangle, \ldots \}$
- $P(\langle \rangle) = 1$, $P(h) = 2$ for all $h \in H \setminus \langle \rangle$
- $u_1(\langle (2,0), y \rangle) = 2$, $u_2(\langle (2,0), y \rangle) = 0$, etc.
Notation:

Let \( h = (a^1, \ldots, a^k) \) be a history, and \( a \) an action.
- Then \( (h, a) \) is the history \( (a^1, \ldots, a^k, a) \).
- If \( h' = (b^1, \ldots, b^l) \), then \( (h, h') \) is the history \( (a^1, \ldots, a^k, b^1, \ldots, b^l) \).
- The set of actions from which player \( P(h) \) can choose after a history \( h \in H \setminus Z \) is written as
  \[
  A(h) = \{ a \mid (h, a) \in H \}.
  \]

Definition (Strategy in an extensive game)

A **strategy** of a player \( i \) in an extensive game \( \Gamma = (N, H, P, (u_i)_{i \in N}) \) is a function \( s_i \) that assigns to each nonterminal history \( h \in H \setminus Z \) with \( P(h) = i \) an action \( a \in A(h) \).

The set of strategies of player \( i \) is denoted as \( S_i \).

**Remark:** Strategies require us to assign actions to histories \( h \), even if it is clear that they will never be played (e.g., because \( h \) will never be reached because of some earlier action).

**Notation (for finite games):** A strategy for a player is written as a string of actions at decision nodes as visited in a breadth-first order.

Example (Strategies in an extensive game)

- Strategies for player 1: \( AE, AF, BE \) and \( BF \)
- Strategies for player 2: \( C \) and \( D \).

Definition (Outcome)

The **outcome** \( O(s) \) of a strategy profile \( s = (s_i)_{i \in N} \) is the (possibly infinite) terminal history \( h = (a^i)^k \), with \( k \in \mathbb{N} \cup \{\infty\} \), such that for all \( \ell \in \mathbb{N} \) with \( 0 \leq \ell < k \),

\[
  s_{P((a^1, \ldots, a^\ell))}((a^1, \ldots, a^\ell)) = a^{\ell+1}.
\]

Example (Outcome)

\[
O(AF, C) = (A, C, F) \quad \text{and} \quad O(AE, D) = (A, D).
\]
3 Solution Concepts

Motivation
Definitions
Solution Concepts
One-Deviation Property
Kuhn's Theorem
Two Extensions
Summary

Nash Equilibria

Motivation
Definitions
Solution Concepts
One-Deviation Property
Kuhn's Theorem
Two Extensions
Summary

Definition (Nash equilibrium in an extensive game)
A Nash equilibrium in an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \) is a strategy profile \( s^* \) such that for every player \( i \in N \) and for all strategies \( s_i \in S_i \),
\[
    u_i(O(s^*_{-i}, s_i)) \geq u_i(O(s^*_{-i}, s_i)).
\]

Induced Strategic Game

Definition (Induced strategic game)
The strategic game \( G \) induced by an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \) is defined by \( G = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle \), where
- \( A'_i = S_i \) for all \( i \in N \), and
- \( u'_i(a) = u_i(O(a)) \) for all \( i \in N \).

Proposition
The Nash equilibria of an extensive game \( \Gamma \) are exactly the Nash equilibria of the induced strategic game \( G \) of \( \Gamma \).

Remarks:
- Each extensive game can be transformed into a strategic game, but the resulting game can be exponentially larger.
- The other direction does not work, because in extensive games, we do not have simultaneous actions.
Empty Threats

Example (Empty threat)

Extensive game:

\[\begin{array}{ccc}
T & B \\
L & 0,0 & 1,2 \\
R & 2,1 & 1,2 \\
\end{array}\]

Strategic form:

\[
\begin{array}{c|cc}
 & L & R \\
T & 0,0 & 2,1 \\
B & 1,2 & 1,2 \\
\end{array}
\]

Nash equilibria: \((B, L)\) and \((T, R)\).

However, \((B, L)\) is not realistic:

- Player 1 plays \(B\), “fearing” response \(L\) to \(T\).
- But player 2 would never play \(L\) in the extensive game.

\(\sim (B, L)\) involves “empty threat”.

Subgames

Idea: Exclude empty threats.

How? Demand that a strategy profile is not only a Nash equilibrium in the strategic form, but also in every subgame.

Definition (Subgame)

A subgame of an extensive game \(\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle\), starting after some history \(h\), is the game \(\Gamma(h) = \langle N, H|_h, P|_h, (u_i|_h)_{i \in N} \rangle\), where

- \(H|_h = \{ h' \mid (h, h') \in H \}\),
- \(P|_h(h') = P(h, h')\) for all \(h' \in H|_h\), and
- \(u_i|_h(h') = u_i(h, h')\) for all \(h' \in H|_h\).

Subgame-Perfect Equilibria

Definition (Subgame-perfect equilibrium)

A strategy profile \(s^*\) in an extensive game \(\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle\) is a subgame-perfect equilibrium if and only if for every player \(i \in N\) and every nonterminal history \(h \in H \setminus Z\) with \(P(h) = i\),

\[
u_i|_h(O_h(s^*_{-i}|_h, s^*_i|_h)) \geq u_i|_h(O_h(s^*_{-i}|_h, s_i))
\]

for every strategy \(s_i \in S_i\) in subgame \(\Gamma(h)\).
**Motivation**

Example (Subgame-perfect equilibria in division game)

Equilibria in subgames:
- \( \Gamma((2,0)) \): \( y \) and \( n \)
- \( \Gamma((1,1)) \): only \( y \)
- \( \Gamma((0,2)) \): only \( y \)
- \( \Gamma(()) \): \((2,0), yyy\)
  - and \((1,1), nyy\)

Nash equilibria (red: empty threat):
- \((2,0), yyy\), \((2,0), yyn\), \((2,0), yny\), \((2,0), ynn\), \((2,0), nny\), \((2,0), nnn\),
- \((1,1), nyy\), \((1,1), nyn\),
- \((0,2), nny\), \((0,2), nnn\).

**Solution**

Subgame-Perfect Equilibria

Two Nash equilibria:
- \((T,R)\): subgame-perfect, because:
  - In history \( h = (T) \): subgame-perfect.
  - In history \( h = () \): player 1 obtains utility 1 when choosing \( B \) and utility of 2 when choosing \( T \).
- \((B,L)\): not subgame-perfect, since \( L \) does not maximize the utility of player 2 in history \( h = (T) \).

**Concepts**

Existence:
- Does every extensive game have a subgame-perfect equilibrium?
- If not, which extensive games do have a subgame-perfect equilibrium?

Computation:
- If a subgame-perfect equilibrium exists, how to compute it?
- How complex is that computation?
Motivation

Positive case (a subgame-perfect equilibrium exists):

- **Step 1:** Show that it suffices to consider local deviations from strategies (for finite-horizon games).
- **Step 2:** Show how to systematically explore such local deviations to find a subgame-perfect equilibrium (for finite games).

Step 1: One-Deviation Property

**Definition**

Let $\Gamma$ be a finite-horizon extensive game. Then $\ell(\Gamma)$ denotes the length of the longest history of $\Gamma$.

**Definition (One-deviation property)**

A strategy profile $s^*$ in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ satisfies the one-deviation property if and only if for every player $i \in N$ and every nonterminal history $h \in H \setminus Z$ with $P(h) = i$,

$$u_i|h(O_h(s^*_{-i}|h, s_i^*)) \geq u_i|h(O_h(s^*_{-i}|h, s_i))$$

for every strategy $s_i \in S_i$ in subgame $\Gamma(h)$ that differs from $s_i^*|h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

**Note:** Without the highlighted parts, this is just the definition of subgame-perfect equilibria!
Step 1: One-Deviation Property

Proof (ctd.)

(⇐) ... WLOG, the number of histories \( h' \) with \( s_i(h') \neq s_i^*(h(h')) \) is at most \( \ell(\Gamma(h)) \) and hence finite (finite horizon assumption!), since deviations not on resulting outcome path are irrelevant.

Illustration: strategies \( s_1^*|_h = AGILN \) and \( s_2^*|_h = CF \) red:

\[
P(h) = 1
\]

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\[
\begin{array}{c}
P(h) = 1 \\
A \\
B \\
C \\
D \\
E \\
F \\
G \\
H \\
I \\
J \\
K \\
L \\
M \\
N \\
O
\end{array}
\]

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Step 1: One-Deviation Property

Proof (ctd.)

(⇐) ... Illustration for WLOG assumption: Assume \( s_1 = BHKMO \) (blue) profitable deviation:

\[
P(h) = 1
\]

Then only \( B \) and \( O \) really matter.

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Step 1: One-Deviation Property

Proof (ctd.)

(⇐) ... Illustration for WLOG assumption: And hence \( \tilde{s}_1 = BGIL (blue) \) also profitable deviation:

\[
P(h) = 1
\]

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Step 1: One-Deviation Property

Proof (ctd.)

(⇐) ...

Choose profitable deviation \( s_i \) in \( \Gamma(h) \) with minimal number of deviation points (such \( s_i \) must exist).

Let \( h' \) be the longest history in \( \Gamma(h) \) with \( s_i(h') \neq s_i^*(h(h')) \),

i.e., “deepest” deviation point for \( s_i \).

Then in \( \Gamma(h, h') \), \( s_i|_{h'} \) differs from \( s_i^*|_{(h, h')} \) only in the initial history.

Moreover, \( s_i|_{h'} \) is a profitable deviation in \( \Gamma(h, h') \), since \( h' \) is the longest history in \( \Gamma(h) \) with \( s_i(h(h')) \neq s_i^*|_{h(h')} \).

So, \( \Gamma(h, h') \) is the desired subgame where a one-step deviation is sufficient to improve utility.

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Step 1: One-Deviation Property

Example

To show that \((AHI, CE)\) is a subgame-perfect equilibrium, it suffices to check these deviating strategies:

Player 1:
- \(G\) in subgame \(\Gamma(\langle A, C \rangle)\)
- \(K\) in subgame \(\Gamma(\langle B, F \rangle)\)
- \(BHI\) in \(\Gamma\)

In particular, e.g., no need to check if strategy \(BGK\) of player 1 is profitable in \(\Gamma\).

Step 2: Kuhn’s Theorem

Theorem (Kuhn)

Every finite extensive game has a subgame-perfect equilibrium.

Proof idea:
- Proof is constructive and builds a subgame-perfect equilibrium bottom-up (aka backward induction).
- For those familiar with the Foundations of AI lecture: generalization of Minimax algorithm to general-sum games with possibly more than two players.

Remark on Infinite-Horizon Games

The corresponding proposition for infinite-horizon games does not hold.

Counterexample (one-player case):

Strategy \(s_i\) with \(s_i(h) = S\) for all \(h \in H \setminus Z\)
- satisfies one deviation property, but
- is not a subgame-perfect equilibrium, since it is dominated by \(s^*_i\) with \(s^*_i(h) = C\) for all \(h \in H \setminus Z\).
Step 2: Kuhn’s Theorem

Example

```
1
(1,5)

A

B

(1,5)

(0,8)

2

(2,3)

D

E

(3,4)

C

F

(0,8)
```

Step 2: Kuhn’s Theorem

**A bit more formally:**

**Proof**

Let $\Gamma = (N, H, P, (u_i)_{i \in N})$ be a finite extensive game. Construct a subgame-perfect equilibrium by induction on $\ell(\Gamma(h))$ for all subgames $\Gamma(h)$. In parallel, construct functions $t_i : H \rightarrow \mathbb{R}$ for all players $i \in N$ s.t. $t_i(h)$ is the payoff for player $i$ in a subgame-perfect equilibrium in subgame $\Gamma(h)$.

**Base case:** If $\ell(\Gamma(h)) = 0$, then $t_i(h) = u_i(h)$ for all $i \in N$.

...
Corresponding proposition for infinite games does not hold.

Counterexamples (both for one-player case):

A) finite horizon, infinite branching factor:

Infinitely many actions $a \in A = [0, 1)$ with payoffs $u_1(\langle a \rangle) = a$ for all $a \in A$.

There exists no subgame-perfect equilibrium in this game.

B) infinite horizon, finite branching factor:

$$u_1(\langle CC \ldots \rangle) = 0 \text{ and } u_1(\langle \underbrace{CCC \ldots}_n S \rangle) = n + 1.$$ 

No subgame-perfect equilibrium.

Uniqueness:

Kuhn's theorem tells us nothing about uniqueness of subgame-perfect equilibria. However, if no two histories get the same evaluation by any player, then the subgame-perfect equilibrium is unique.
Pirates: General Setting & Utility

1. The pirates do not trust each other, and will neither make nor honor any promises between pirates apart from a proposed distribution plan that gives a whole number of gold coins to each pirate.

2. Pirates base their decisions on three factors. First of all, each pirate wants to survive. Second, everything being equal, each pirate wants to maximize the number of gold coins each receives. Third, each pirate would prefer to throw another overboard, if all other results would otherwise be equal.

Pirates: Formalization

- Players $N = \{A, B, C, D, E\}$;
- actions are:
  - proposals by a pirate: $\langle A : x_A, B : x_B, C : x_C, D : x_D, E : x_E \rangle$, with $\sum_{i \in \{A, B, C, D, E\}} x_i = 100$;
  - votings: $y$ for accepting, $n$ for rejecting;
- histories are sequences of a proposal, followed by votings of the alive pirates;
- utilities:
  - for pirates who are alive: utilities are according to the accepted proposal plus $x/100$, $x$ being the number of dead pirates;
  - for dead pirates: $-100$.

Remark: Very large game tree!

Pirates: Analysis by Backward Induction

1. Assume only $D$ and $E$ are still alive. $D$ can propose $\langle A : 0, B : 0, C : 0, D : 100, E : 0 \rangle$, because $D$ has the casting vote!

2. Assume $C$, $D$, and $E$ are alive. For $C$ it is enough to offer 1 coin to $E$ to get his vote: $\langle A : 0, B : 0, C : 99, D : 0, E : 1 \rangle$.

3. Assume $B$, $C$, $D$, and $E$ are alive. $B$ offering $D$ one coin is enough because of the casting vote: $\langle A : 0, B : 99, C : 0, D : 1, E : 0 \rangle$.

4. Assume $A$, $B$, $C$, $D$, and $E$ are alive. $A$ offering $C$ and $E$ each one coin is enough: $\langle A : 98, B : 0, C : 1, D : 0, E : 1 \rangle$ (note that giving 1 to $D$ instaed to $E$ does not help).

6 Two Extensions

- Simultaneous Moves
- Chance
Simultaneous Moves

Definition
An extensive game with simultaneous moves is a tuple
\( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \), where
- \( N, H, P \) and \( (u_i) \) are defined as before, and
- \( P : H \rightarrow 2^N \) assigns to each nonterminal history a set of players to move; for all \( h \in H \setminus Z \), there exists a family \( (A_i(h))_{i \in P(h)} \) such that
  \[ A(h) = \{ a \mid (h, a) \in H \} = \prod_{i \in P(h)} A_i(h). \]

Remark:
- The one-deviation property still holds for extensive game with perfect information and simultaneous moves.
- Kuhn’s theorem does not hold for extensive game with simultaneous moves.

Example: Matching Pennies can be viewed as extensive game with simultaneous moves. No Nash equilibrium/subgame-perfect equilibrium.

\[
\begin{array}{c|cc}
  & H & T \\
\hline
H & 1, -1 & -1, 1 \\
T & -1, 1 & 1, -1 \\
\end{array}
\]

\( \Rightarrow \) Need more sophisticated solution concepts (cf. mixed strategies). Not covered in this lecture.
**Subgame-perfect equilibria (ctd.):**

Example: Three-Person Cake Splitting Game

Simultaneous Moves

Subgame-perfect equilibria:

- Subgames after legal split \((x_1, x_2, x_3)\) by player 1:
  - NE \((y, y)\) (both accept)
  - NE \((n, n)\) (neither accepts)
  - If \(x_2 = 0\), NE \((n, y)\) (only player 3 accepts)
  - If \(x_3 = 0\), NE \((y, n)\) (only player 2 accepts)

---

**Simultaneous Moves**

Example: Three-Person Cake Splitting Game

Simultaneous Moves

**Formally:**

\[
\begin{align*}
N &= \{1, 2, 3\} \\
X &= \{(x_1, x_2, x_3) \in [0, 1]^3 | x_1 + x_2 + x_3 = 1\} \\
H &= \{()\} \cup \{(x) | x \in X\} \cup \{(x, z) | x \in X, z \in \{y, n\} \times \{y, n\}\} \\
P(\{\}\) &= \{1\} \\
P(\{x\}) &= \{2, 3\} \text{ for all } x \in X \\
u_i((x, z)) &= \begin{cases} 
0 & \text{if } z \in \{(y, n), (n, y), (n, n)\} \\
x_i & \text{if } z = (y, y).
\end{cases} \text{ for all } i \in N
\end{align*}
\]

---

**Simultaneous Moves**

Example: Three-Person Cake Splitting Game

Subgame-perfect equilibria (ctd.):

- **Entire game:**
  Let \(s_2\) and \(s_3\) be any two strategies of players 2 and 3 such that for all splits \(x \in X\) the profile \((s_2((x)), s_3((x)))\) is one of the NEs from above. Let \(X_y = \{x \in X | s_2((x)) = s_3((x)) = y\}\) be the set of splits accepted under \(s_2\) and \(s_3\). Distinguish three cases:
  - \(X_y = \emptyset\) or \(x_1 = 0\) for all \(x \in X_y\). Then \((s_1, s_2, s_3)\) is a subgame-perfect equilibrium for any possible \(s_1\).
  - \(X_y \neq \emptyset\) and there are splits \(x_{\text{max}} = (x_1, x_2, x_3) \in X_y\) that maximize \(x_1 > 0\). Then \((s_1, s_2, s_3)\) is a subgame-perfect equilibrium if and only if \(s_1((\))\) is such a split \(x_{\text{max}}\).
  - \(X_y \neq \emptyset\) and there are no splits \((x_1, x_2, x_3) \in X_y\) that maximize \(x_1\). Then there is no subgame-perfect equilibrium, in which player 2 follows strategy \(s_2\) and player 3 follows strategy \(s_3\).

---

**Chance Moves**

Definition

An extensive game with chance moves is a tuple \(\Gamma = (N, H, P, f_c, (u_i)_{i \in N})\), where

- \(N, A, H\) and \(u_i\) are defined as before,
- the player function \(P : H \setminus Z \to N \cup \{c\}\) can also take the value \(c\) for a chance node, and
- for each \(h \in H \setminus Z\) with \(P(h) = c\), the function \(f_c(\cdot | h)\) is a probability distribution on \(A(h)\) such that the probability distributions for all \(h \in H\) are independent of each other.
Intended meaning of chance moves: In chance node, an applicable action is chosen randomly with probability according to $f_c$.

Strategies: Defined as before.

Outcome: For a given strategy profile, the outcome is a probability distribution on the set of terminal histories.

Payoffs: For player $i$, $U_i$ is the expected payoff (with weights according to outcome probabilities).

**Example**

```
\[ P(\langle \rangle) = 1 \]
\[ P(\langle A \rangle) = \frac{1}{3} \]
\[ P(\langle B \rangle) = \frac{1}{3} \]
\[ P(\langle B, F \rangle) = \frac{1}{3} \]
```

**Remark:**
The one-deviation property and Kuhn’s theorem still hold in the presence of chance moves. When proving Kuhn’s theorem, expected utilities have to be used.
For finite-horizon extensive games, it suffices to consider local deviations when looking for better strategies.

For infinite-horizon games, this is not true in general.

Every finite extensive game has a subgame-perfect equilibrium.

This does not generally hold for infinite games, no matter is game is infinite due to infinite branching factor or infinitely long histories (or both).

With chance moves, one deviation property and Kuhn's theorem still hold.

With simultaneous moves, Kuhn's theorem no longer holds.