Motivation

- **So far:** All players move simultaneously, and then the outcome is determined.
- **Often in practice:** Several moves in sequence (e.g. in chess).
  \[\rightarrow\] cannot be directly reflected by strategic games.
- **Extensive games** (with perfect information) reflect such situations by modeling games as game trees.
- **Idea:** Players have several decision points where they can decide how to play.
- **Strategies:** Mappings from decision points in the game tree to actions to be played.
Extensive Games

Definition (Extensive game with perfect information)

An extensive game with perfect information is a tuple \( \Gamma = (N,H,P,\langle u_i \rangle_{i \in N}) \) that consists of:

- A finite non-empty set \( N \) of players.
- A set \( H \) of (finite or infinite) sequences, called histories, such that
  - it contains the empty sequence \( \langle \rangle \in H \),
  - \( H \) is closed under prefixes; if \( \langle a^1,\ldots,a^k \rangle \in H \) for some \( k \in \mathbb{N} \cup \{\infty\} \), and \( l < k \), then also \( \langle a^1,\ldots,a^l \rangle \in H \), and
  - \( H \) is closed under limits; if for some infinite sequence \( \langle a^i \rangle_{i=1}^{\infty} \), we have \( \langle a^k \rangle_{k=1}^{\infty} \in H \) for all \( k \in \mathbb{N} \), then \( \langle a^i \rangle_{i=1}^{\infty} \in H \).
- All infinite histories and all histories \( \langle a^i \rangle_{i=1}^{k} \in H \), for which there is no \( a^{k+1} \) such that \( \langle a^i \rangle_{i=1}^{k+1} \in H \) are called terminal histories \( Z \). Components of a history are called actions.

Example (Division game)

- Two identical objects should be divided among two players.
- Player 1 proposes an allocation.
- Player 2 agrees or rejects.
  - On agreement: Allocation as proposed.
  - On rejection: Nobody gets anything.

\[
\begin{align*}
P(\langle \rangle) &= 1 \\
P(\langle(2,0)\rangle) &= 2 \\
P(\langle(1,1)\rangle) &= 2 \\
P(\langle(0,2)\rangle) &= 2 \\
(2,0) &\quad n \\
(1,1) &\quad y \\
(0,0) &\quad y \\
(0,2) &\quad n \\
(0,0) &\quad y \\
\end{align*}
\]

Extensive Games

Definition (Extensive game with perfect information, cttd.)

- A player function \( P : H \setminus Z \rightarrow N \) that determines which player’s turn it is to move after a given nonterminal history.
- For each player \( i \in N \), a utility function (or payoff function) \( u_i : Z \rightarrow \mathbb{R} \) defined on the set of terminal histories.

The game is called finite, if \( H \) is finite. It has a finite horizon, if the length of histories is bounded from above.

Assumption: All ingredients of \( \Gamma \) are common knowledge amongst the players of the game.

Terminology: In the following, we will simply write extensive games instead of extensive games with perfect information.

Example (Division game, formally)

- \( N = \{1,2\} \)
- \( H = \{\langle \rangle, \langle(2,0)\rangle, \langle(1,1)\rangle, \langle(0,2)\rangle, \langle(2,0),y\rangle, \langle(2,0),n\rangle, \ldots\} \)
- \( P(\langle \rangle) = 1, P(h) = 2 \) for all \( h \in H \setminus Z \) with \( h \neq \langle \rangle \)
- \( u_1(\langle(2,0),y\rangle) = 2, u_2(\langle(2,0),y\rangle) = 0, \) etc.
**Strategies**

**Definition (Strategy in an extensive game)**
A strategy of a player $i$ in an extensive game $\Gamma = (N, H, P, (u_i)_{i \in N})$ is a function $s_i$ that assigns to each nonterminal history $h \in H \setminus Z$ an action $a \in A(h)$. The set of strategies of player $i$ is denoted as $S_i$.

**Remark:** Strategies require us to assign actions to histories $h$, even if it is clear that they will never be played (e.g., because $h$ will never be reached because of some earlier action).

**Notation (for finite games):** A strategy for a player is written as a string of actions at decision nodes as visited in a breadth-first order.

**Example (Strategies in an extensive game)**

- Strategies for player 1: $AE$, $AF$, $BE$ and $BF$
- Strategies for player 2: $C$ and $D$.

**Outcome**

**Definition (Outcome)**
The outcome $O(s)$ of a strategy profile $s = (s_i)_{i \in N}$ is the (possibly infinite) terminal history $h = \langle a_i \rangle_{i=1}^k$, with $k \in \mathbb{N} \cup \{\infty\}$, such that for all $\ell \in \mathbb{N}$ with $0 \leq \ell < k$,

$$s_{P(\langle a_1, \ldots, a_\ell \rangle)}(\langle a_1, \ldots, a_\ell \rangle) = a_{\ell+1}.$$
Nash Equilibria

Definition (Nash equilibrium in an extensive game)
A Nash equilibrium in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is a strategy profile $s^*$ such that for every player $i \in N$ and for all strategies $s_i \in S_i$,

$$u_i(O(s^*-i, s_i^*)) \geq u_i(O(s^*-i, s_i)).$$

Induced Strategic Game

Definition (Induced strategic game)
The strategic game $G$ induced by an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is defined by $G = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$, where

- $A'_i = S_i$ for all $i \in N$,
- $u'_i(a) = u_i(O(a))$ for all $i \in N$.

Proposition
The Nash equilibria of an extensive game $\Gamma$ are exactly the Nash equilibria of the induced strategic game $G$ of $\Gamma$.

Remarks:
- Each extensive game can be transformed into a strategic game, but the resulting game can be exponentially larger.
- The other direction does not work, because in extensive games, we do not have simultaneous actions.
Empty Threats

Example (Empty threat)

**Extensive game:**

- **P(()) = 1**
- **P(T) = 2**

**Strategic form:**

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T</strong></td>
<td>0.0</td>
<td>2.1</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>1.2</td>
<td>1.2</td>
</tr>
</tbody>
</table>

**Nash equilibria:** (B, L) and (T, R).

However, (B, L) is not realistic:
- Player 1 plays B, "fearing" response L to T.
- But player 2 would never play L in the extensive game.

\( \Rightarrow (B, L) \) involves "empty threat".

Subgames

**Definition (Subgame)**

A subgame of an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \), starting after history \( h \), is the game \( \Gamma(h) = \langle N, H|h, P|_h, (u|_h)_{i \in N} \rangle \), where:

- \( H|h = \{ h' \mid (h, h') \in H \} \),
- \( P|_h(h') = P(h, h') \) for all \( h' \in H|h \), and
- \( u|_h(h') = u_i(h, h') \) for all \( h' \in H|h \).

**Subgame-Perfect Equilibria**

**Definition (Subgame-perfect equilibrium)**

A strategy profile \( s^* \) in an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \) is a subgame-perfect equilibrium if and only if for every player \( i \in N \) and every nonterminal history \( h \in H \setminus Z \) with \( P(h) = i \),

\[
 u_i|_h(O_h(s^*|_h, s^*_i|_h)) \geq u_i|_h(O_h(s^*|_h, s_i))
\]

for every strategy \( s_i \in S_i \) in subgame \( \Gamma(h) \).
Two Nash equilibria:
- \((T, R)\): subgame-perfect, because:
  - In history \(h = \langle T \rangle\): subgame-perfect.
  - In history \(h = \langle \rangle\): player 1 obtains utility 1 when choosing \(B\) and utility of 2 when choosing \(T\).
- \((B, L)\): not subgame-perfect, since \(L\) does not maximize the utility of player 2 in history \(h = \langle T \rangle\).

Nash equilibria (red: empty threat):
- \((2, 0), yyy\), \((2, 0), yyn\), \((2, 0), yny\), \((2, 0), ynn\), \((2, 0), nny\), \((2, 0), nnn\),
- \((1, 1), nyy\), \((1, 1), nyn\).
- \((0, 2), nny\)

Subgame-Perfect Equilibria

Example (Subgame-perfect equilibria in division game)

Equilibria in subgames:
- in \(\Gamma((2, 0))\): \(y\) and \(n\)
- in \(\Gamma((1, 1))\): only \(y\)
- in \(\Gamma((0, 2))\): only \(y\)
- in \(\Gamma(\langle \rangle)\): \((2, 0), yyy\) and \((1, 1), nyy\)

4 One-Deviation Property

- Existence:
  - Does every extensive game have a subgame-perfect equilibrium?
  - If not, which extensive games do have a subgame-perfect equilibrium?
- Computation:
  - If a subgame-perfect equilibrium exists, how to compute it?
  - How complex is that computation?
Step 1: One-Deviation Property

Definition

A strategy profile $s^*$ in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ satisfies the one-deviation property if and only if for every player $i \in N$ and every nonterminal history $h \in H \setminus Z$ with $P(h) = i$,

$$u_i|h(O_h(s^*|h, s^*_i|h)) \geq u_i|h(O_h(s^*_i|h, s_i))$$

for every strategy $s_i \in S_i$ in subgame $\Gamma(h)$ that differs from $s^*_i|h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

Note: Without the highlighted parts, this is just the definition of subgame-perfect equilibria!
Step 1: One-Deviation Property

Proof (ctd.)

Illustration: strategies $s_1^*|_h = AGILN$ and $s_2^*|_h = CF$ red:

Then only $B$ and $O$ really matter.

Proof (ctd.)

Illustration for WLOG assumption: And hence profitable deviation:

Choose profitable deviation $s_i$ in $\Gamma(h)$ with minimal number of deviation points (such $s_i$ must exist).

Let $h^*$ be the longest history in $\Gamma(h)$ with $s_i(h^*) \neq s_i^*|_h(h^*)$, i.e., “deepest” deviation point for $s_i$.

Then in $\Gamma(h, h^*)$, $s_i|_{h^*}$ differs from $s_i^*|_{(h, h^*)}$ only in the initial history.

Moreover, $s_i|_{h^*}$ is a profitable deviation in $\Gamma(h, h^*)$, since $h^*$ is the longest history in $\Gamma(h)$ with $s_i(h^*) \neq s_i^*|_{(h, h^*)}$.

So, $\Gamma(h, h^*)$ is the desired subgame where a one-step deviation is sufficient to improve utility.
Step 1: One-Deviation Property

Example

To show that \((AHI, CE)\) is a subgame-perfect equilibrium, it suffices to check these deviating strategies:

**Player 1:**
- \(G\) in subgame \(\Gamma(\langle A, C \rangle)\)
- \(K\) in subgame \(\Gamma(\langle B, F \rangle)\)
- \(BHI\) in \(\Gamma\)

**Player 2:**
- \(D\) in subgame \(\Gamma(\langle A \rangle)\)
- \(F\) in subgame \(\Gamma(\langle B \rangle)\)

In particular, e.g., no need to check if strategy \(BGK\) of player 1 is profitable in \(\Gamma\).

Remark on Infinite-Horizon Games

The corresponding proposition for infinite-horizon games does not hold.

Counterexample (one-player case):

Strategy \(s_i\) with \(s_i(h) = S\) for all \(h \in H \setminus Z\)
- satisfies one deviation property, but
- is not a subgame-perfect equilibrium, since it is dominated by \(s'_i\) with \(s'_i(h) = C\) for all \(h \in H \setminus Z\).

Step 2: Kuhn’s Theorem

**Theorem (Kuhn)**
Every finite extensive game has a subgame-perfect equilibrium.

**Proof idea:**
- Proof is constructive and builds a subgame-perfect equilibrium bottom-up (aka backward induction).
- For those familiar with the Foundations of AI lecture: generalization of Minimax algorithm to general-sum games with possibly more than two players.
Step 2: Kuhn’s Theorem

Example

Motivation
Definitions
Solution
Concepts
One-Deviation
Property
Kuhn’s
Theorem
Two
Extensions
Summary

\[ s_2(\langle A \rangle) = C \]  \[ t_1(\langle A \rangle) = 1 \]  \[ t_2(\langle A \rangle) = 5 \]

\[ s_2(\langle B \rangle) = F \]  \[ t_1(\langle B \rangle) = 0 \]  \[ t_2(\langle B \rangle) = 8 \]

\[ s_1(\langle \emptyset \rangle) = A \]  \[ t_1(\langle \emptyset \rangle) = 1 \]  \[ t_2(\langle \emptyset \rangle) = 5 \]

Inductive case: If \( t_i(h) \) already defined for all \( h \in H \) with \( \ell(\Gamma(h)) \leq k \), consider \( h^* \in H \) with \( \ell(\Gamma(h^*)) = k + 1 \) and \( P(h^*) = i \).
For all \( a \in A(h^*), \) \( \ell(\Gamma(h^*, a)) \leq k \), let
\[
\begin{align*}
  s_i(h^*) & := \arg\max_{a \in A(h^*)} t_i(h^*, a) \quad \text{and} \\
  t_j(h^*) & := t_j(h^*, s_i(h^*)) \quad \text{for all players } j \in N.
\end{align*}
\]

Inductively, we obtain a strategy profile \( s \) that satisfies the one-deviation property.
With the one-deviation property lemma it follows that \( s \) is a subgame-perfect equilibrium.
Step 2: Kuhn’s Theorem

Remark on Infinite Games

Corresponding proposition for infinite games does not hold.

Counterexamples (both for one-player case):

A) finite horizon, infinite branching factor:

Infinitely many actions \( a \in A = [0, 1) \) with payoffs \( u_1(\langle a \rangle) = a \) for all \( a \in A \).
There exists no subgame-perfect equilibrium in this game.

B) infinite horizon, finite branching factor:

\[
\begin{array}{ccccccc}
S & S & S & S & S & S & C, \ldots, 0 \\
1 & 2 & 3 & 4 & 5 & 6 & \\
\end{array}
\]

\[ u_1(\langle C \rangle \ldots) = 0 \text{ and } u_1(\langle C \rangle \ldots \langle C \rangle \ldots) = n + 1. \]

No subgame-perfect equilibrium.

Uniqueness:

Kuhn's theorem tells us nothing about uniqueness of subgame-perfect equilibria. However, if no two histories get the same evaluation by any player, then the subgame-perfect equilibrium is unique.

Extended Example: Pirate Game

There are 5 rational pirates, \( A, B, C, D \) and \( E \). They find 100 gold coins. They must decide how to distribute them.

The pirates have a strict order of seniority: \( A \) is senior to \( B \), who is senior to \( C \), who is senior to \( D \), who is senior to \( E \).

The pirate world's rules of distribution say that the most senior pirate first proposes a distribution of coins. The pirates, including the proposer, then vote on whether to accept this distribution (in order from most junior to senior). In case of a tie vote, the proposer has the casting vote. If the distribution is accepted, the coins are disbursed and the game ends. If not, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to apply the method again.
The pirates do not trust each other, and will neither make nor honor any promises between pirates apart from a proposed distribution plan that gives a whole number of gold coins to each pirate.

Pirates base their decisions on three factors. First of all, each pirate wants to survive. Second, everything being equal, each pirate wants to maximize the number of gold coins each receives. Third, each pirate would prefer to throw another overboard, if all other results would otherwise be equal.

Players \( N = \{A, B, C, D, E\} \);

actions are:

- proposals by a pirate: \( \langle A : x_A, B : x_B, C : x_C, D : x_D, E : x_E \rangle \), with \( \sum_{i \in \{A, B, C, D, E\}} x_i = 100 \);
- votings: \( y \) for accepting, \( n \) for rejecting;

histories are sequences of a proposal, followed by votings of the alive pirates;

utilities:
- for pirates who are alive: utilities are according to the accepted proposal plus \( x/100 \), \( x \) being the number of dead pirates;
- for dead pirates: -100.

Remark: Very large game tree!

Assume only \( D \) and \( E \) are still alive. \( D \) can propose \( \langle A : 0, B : 0, C : 0, D : 100, E : 0 \rangle \), because \( D \) has the casting vote!

Assume \( C, D, \) and \( E \) are alive. For \( C \) it is enough to offer 1 coin to \( E \) to get his vote: \( \langle A : 0, B : 0, C : 99, D : 0, E : 1 \rangle \).

Assume \( B, C, D, \) and \( E \) are alive. \( B \) offering \( D \) one coin is enough because of the casting vote: \( \langle A : 0, B : 99, C : 0, D : 1, E : 0 \rangle \).

Assume \( A, B, C, D, \) and \( E \) are alive. \( A \) offering \( C \) and \( E \) each one coin is enough: \( \langle A : 98, B : 0, C : 1, D : 0, E : 1 \rangle \) (note that giving 1 to \( D \) instaed to \( E \) does not help).
Simultaneous Moves

**Definition**

An extensive game with simultaneous moves is a tuple 
\[ \Gamma = (N, H, P, (u_i)_{i \in N}) \], where
- \( N \), \( H \), \( P \) and \((u_i)_{i \in N}\) are defined as before, and
- \( P : H \to 2^N \) assigns to each nonterminal history a set of players to move; for all \( h \in H \setminus Z \), there exists a family \((A_i(h))_{i \in P(h)}\) such that
  \[ A(h) = \{ a \mid (h, a) \in H \} = \prod_{i \in P(h)} A_i(h). \]

**Remark:**

- The one-deviation property still holds for extensive game with perfect information and simultaneous moves.
- Kuhn's theorem does not hold for extensive game with simultaneous moves.

**Example:** Matching Pennies can be viewed as extensive game with simultaneous moves. No Nash equilibrium/subgame-perfect equilibrium.

\[
\begin{array}{c|cc}
& H & T \\
\hline
H & 1, -1 & -1, 1 \\
T & -1, 1 & 1, -1 \\
\end{array}
\]

\[\Rightarrow\] Need more sophisticated solution concepts (cf. mixed strategies). Not covered in this lecture.

Simultaneous Moves

**Intended meaning of simultaneous moves:** All players from \( P(h) \) move simultaneously.

**Strategies:** Functions \( s_i : h \mapsto a_i \) with \( a_i \in A_i(h) \).

**Histories:** Sequences of vectors of actions.

**Outcome:** Terminal history reached when tracing strategy profile.

**Payoffs:** Utilities at outcome history.

Simultaneous Moves

Example: Three-Person Cake Splitting Game

**Setting:**

- Three players have to split a cake fairly.
- Player 1 suggest split: shares \( x_1, x_2, x_3 \in [0, 1] \) s.t. \( x_1 + x_2 + x_3 = 1 \).
- Then players 2 and 3 simultaneously and independently decide whether to accept ("y") or reject ("n") the suggested splitting.
- If both accept, each player \( i \) gets his allotted share (utility \( x_i \)). Otherwise, no player gets anything (utility 0).
Formally:

\[
N = \{1, 2, 3\} \\
X = \{(x_1, x_2, x_3) \in [0, 1]^3 | x_1 + x_2 + x_3 = 1\} \\
H = \{()\} \cup \{(x) | x \in X\} \cup \{\langle x, z \rangle | x \in X, z \in \{y, n\} \times \{y, n\}\} \\
P(\langle\rangle) = \{1\} \\
P(\langle x \rangle) = \{2, 3\} \text{ for all } x \in X \\
u_i(\langle x, z \rangle) = \begin{cases} 
0 & \text{if } z \in \{(y, n), (n, y), (n, n)\} \\
x_i & \text{if } z = (y, y).
\end{cases} \text{ for all } i \in N
\]
**Chance Moves**

- **Intended meaning of chance moves:** In chance node, an applicable action is chosen randomly with probability according to $f_c$.
- **Strategies:** Defined as before.
- **Outcome:** For a given strategy profile, the outcome is a probability distribution on the set of terminal histories.
- **Payoffs:** For player $i$, $U_i$ is the expected payoff (with weights according to outcome probabilities).

---

**Example**

```
\begin{align*}
P() &= 1, (2, 3) \\
P(A) &= c, (1, 4) \\
P(B) &= c, (2, 3) \\
f_c(D|A) &= \frac{1}{2}, E \\
f_c(F|B) &= \frac{1}{3}, F \\
f_c(G|B) &= \frac{2}{3}, G \\
\end{align*}
```

---

**Remark:**
The one-deviation property and Kuhn's theorem still hold in the presence of chance moves. When proving Kuhn’s theorem, *expected* utilities have to be used.
Summary

- For finite-horizon extensive games, it suffices to consider local deviations when looking for better strategies.
- For infinite-horizon games, this is not true in general.
- Every finite extensive game has a subgame-perfect equilibrium.
- This does not generally hold for infinite games, no matter is game is infinite due to infinite branching factor or infinitely long histories (or both).

- With chance moves, one deviation property and Kuhn's theorem still hold.
- With simultaneous moves, Kuhn's theorem no longer holds.