### Strategic Games

**Definition (Strategic game)**

A strategic game is a tuple $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ where

- a nonempty finite set $N$ of players,
- for each player $i \in N$, a nonempty set $A_i$ of actions (or strategies), and
- for each player $i \in N$, a payoff function $u_i : A \to \mathbb{R}$, where $A = \prod_{i \in N} A_i$.  

A strategic game $G$ is called finite if $A$ is finite.

A strategy profile is a tuple $a = (a_1, \ldots, a_{|N|}) \in A$.

We can describe finite strategic games using payoff matrices.

**Example:** Two-player game where player 1 has actions $T$ and $B$, and player 2 has actions $L$ and $R$, with payoff matrix

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$w_1, w_2$</td>
<td>$x_1, x_2$</td>
</tr>
<tr>
<td>$B$</td>
<td>$y_1, y_2$</td>
<td>$z_1, z_2$</td>
</tr>
</tbody>
</table>

**Read:** If player 1 plays $T$ and player 2 plays $L$ then player 1 gets payoff $w_1$ and player 2 gets payoff $w_2$, etc.
Prisoner’s Dilemma

Example (Prisoner’s Dilemma (informally))

Two prisoners are interrogated separately, and have the options to either cooperate (C) with their fellow prisoner and stay silent, or defect (D) and accuse the fellow prisoner of the crime.

Possible outcomes:
- Both cooperate: no hard evidence against either of them, only short prison sentences for both.
- One cooperates, the other defects: the defecting prisoner is set free immediately, and the cooperating prisoner gets a very long prison sentence.
- Both confess: both get medium-length prison sentences.

Hawk and Dove

An anti-coordination game:

Example (Hawk and Dove (informally))

In a fight for resources two players can behave either like a dove (D), yielding, or like a hawk (H), attacking.

Possible outcomes:
- Both players behave like doves: both players share the benefit.
- A hawk meets a dove: the hawk wins and gets the bigger part.
- Both players behave like hawks: the benefit gets lost completely because they will fight each other.
Matching Pennies

A strictly competitive game:

**Example (Matching Pennies (informally))**

Two players can choose either heads ($H$) or tails ($T$) of a coin.

Possible outcomes:
- Both players make the same choice: player 1 receives one Euro from player 2.
- The players make different choices: player 2 receives one Euro from player 1.

Matching Pennies

**Example (Matching Pennies (payoff matrix))**

Strategies $A_1 = A_2 = \{H, T\}$.

<table>
<thead>
<tr>
<th></th>
<th>$H$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>$1,-1$</td>
<td>$-1,1$</td>
</tr>
<tr>
<td>$T$</td>
<td>$-1,1$</td>
<td>$1,-1$</td>
</tr>
</tbody>
</table>

Bach or Stravinsky (aka Battle of the Sexes)

A coordination game:

**Example (Bach or Stravinsky (informally))**

Two persons, one of whom prefers Bach whereas the other prefers Stravinsky want to go to a concert together. For both it is more important to go to the same concert than to go to their favorite one. Let $B$ be the action of going to the Bach concert and $S$ the action of going to the Stravinsky concert.

Possible outcomes:
- Both players make the same choice: the player whose preferred option is chosen gets high payoff, the other player gets medium payoff.
- The players make different choices: they both get zero payoff.

Bach or Stravinsky (aka Battle of the Sexes)

**Example (Bach or Stravinsky (payoff matrix))**

Strategies $A_1 = A_2 = \{B, S\}$.

<table>
<thead>
<tr>
<th></th>
<th>$B$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>$2,1$</td>
<td>$0,0$</td>
</tr>
<tr>
<td>$S$</td>
<td>$0,0$</td>
<td>$1,2$</td>
</tr>
</tbody>
</table>
**Congestion Game**

**Example (A congestion game)**

![Congestion Game Diagram]

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>player 1</td>
<td>-2, -2</td>
<td>-1.5, -1.5</td>
<td>-2, -1.5</td>
</tr>
<tr>
<td>player 2</td>
<td>-1.5, -1.5</td>
<td>-2, -2</td>
<td>-2, -1.5</td>
</tr>
<tr>
<td></td>
<td>-1.5, -2</td>
<td>-1.5, -2</td>
<td>-2, -2</td>
</tr>
</tbody>
</table>

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**2 Solution Concepts and Notation**

**Question:** What is a “solution” of a strategic game?

**Answer:**
- A strategy profile where all players play strategies that are rational (i.e., in some sense optimal).
- **Note:** There are different ways of making the above item precise (different solution concepts).
- A **solution concept** is a formal rule for predicting how a game will be played.

In the following, we will consider some solution concepts:
- Iterated dominance
- Nash equilibrium
- (Subgame-perfect equilibrium)

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**Solution Concepts and Notation**

**Notation:** we want to write down strategy profiles where one player’s strategy is removed or replaced.

Let \( a = (a_1, \ldots, a_N) \in A = \prod_{i \in N} A_i \) be a strategy profile.

We write:
- \( A_{-i} := \prod_{j \neq i} A_j \),
- \( a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N) \), and
- \( (a_{-i}, a_i') := (a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_N) \).

**Example**

Let \( A_1 = \{T, B\}, A_2 = \{L, R\}, A_3 = \{X, Y, Z\}, \) and \( a := (T, R, Z) \).

Then \( a_{-1} = (R, Z), a_{-2} = (T, Z), a_{-3} = (T, R) \).

Moreover, \( (a_{-2}, L) = (T, L, Z) \).

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**Solution Concepts and Notation**

**Notation:** we want to write down strategy profiles where one player’s strategy is removed or replaced.

Let \( a = (a_1, \ldots, a_N) \in A = \prod_{i \in N} A_i \) be a strategy profile.

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- \( (a_{-i}, a_i') := (a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_N) \).

**Example**

Let \( A_1 = \{T, B\}, A_2 = \{L, R\}, A_3 = \{X, Y, Z\}, \) and \( a := (T, R, Z) \).

Then \( a_{-1} = (R, Z), a_{-2} = (T, Z), a_{-3} = (T, R) \).

Moreover, \( (a_{-2}, L) = (T, L, Z) \).

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Strictly Dominated Strategies

Definition (Strictly dominated strategy)
Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic game.

A strategy $a_i \in A_i$ is called strictly dominated in $G$ if there is a strategy $a_i^* \in A_i$ such that for all strategy profiles $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^*).$$

We say that $a_i^*$ strictly dominates $a_i$.

If $a_i^* \in A_i$ strictly dominates every other strategy $a_i' \in A_i \setminus \{a_i^*\}$, we call $a_i^*$ strictly dominant in $G$.

Remark: Playing strictly dominated strategies is irrational.

This suggests a solution concept:
iterative elimination of strictly dominated strategies:

while some strictly dominated strategy is left:
   eliminate some strictly dominated strategy
if a unique strategy profile remains:
   this unique profile is the solution
Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for the prisoner’s dilemma)

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
</tr>
<tr>
<td>player 1</td>
<td>3,3</td>
</tr>
<tr>
<td></td>
<td>4,0</td>
</tr>
</tbody>
</table>

- Step 1: eliminate row C (strictly dominated by row D)
- Step 2: eliminate column C (strictly dominated by col. D)
### Strictly Dominated Strategies

**Example (Iterative elim. of strictly dominated strategies)**

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
</tr>
<tr>
<td><strong>T</strong></td>
<td>2,1</td>
</tr>
<tr>
<td><strong>M</strong></td>
<td>1,2</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>0,0</td>
</tr>
</tbody>
</table>

- **Step 1:** eliminate row B (strictly dominated by row M)
- **Step 2:** eliminate column R (strictly dominated by col. L)

**Step 1:** eliminate row B (strictly dominated by row M)
**Step 2:** eliminate column R (strictly dominated by col. L)
**Step 3:** eliminate row M (strictly dominated by row T)
Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for Bach or Stravinsky)

Stravinsky enthusiast

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>S</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

Bach enthusiast

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>S</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

- No strictly dominated strategies.
- All strategies survive iterative elimination of strictly dominated strategies.
- All strategies rationalizable.

Remark
Strict dominance between actions is rather rare. We should identify more constraints on “solutions”, better solution concepts.

Proposition
The result of iterative elimination of strictly dominated strategies is unique, i.e., independent of the elimination order.

Proof.
Homework.
Weakly Dominated Strategies

**Definition (Weakly dominated strategy)**

Let \( G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \) be a strategic game.

A strategy \( a_i \in A_i \) is called **weakly dominated in** \( G \) if there is a strategy \( a_i^+ \in A_i \) such that for all profiles \( a_{-i} \in A_{-i} \),

\[
u_i(a_{-i}, a_i) \leq u_i(a_{-i}, a_i^+) \]

and that for at least one profile \( a_{-i} \in A_{-i} \),

\[
u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).\]

We say that \( a_i^+ \) weakly dominates \( a_i \).

If \( a_i^+ \in A_i \) weakly dominates every other strategy \( a_i' \in A_i \setminus \{a_i^+\} \),
we call \( a_i^+ \) weakly dominant in \( G \).

---

**Example (Iterative elimination of weakly dominated strategies)**

<table>
<thead>
<tr>
<th></th>
<th>( L )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T</strong></td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td><strong>M</strong></td>
<td>2,1</td>
<td>1,1</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>0,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

**Step 1:** eliminate row \( B \) (weakly dominated by row \( M \)).

\( u_1(M, L) = 2 > 0 = u_1(B, L) \) and \( u_1(M, R) = 1 = u_1(B, R) \)
Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

<table>
<thead>
<tr>
<th>player 1</th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>2, 1</td>
</tr>
<tr>
<td>M</td>
<td>2, 1</td>
</tr>
</tbody>
</table>

- Step 1: eliminate row $B$ (weakly dominated by row $M$), $u_1(M, L) = 2 > 0 = u_1(B, L)$ and $u_1(M, R) = 1 = u_1(B, R)$
- Step 2: eliminate column $R$ (weakly dominated by col. $L$)

Iterative elimination of weakly dominated strategies:
- leads to smaller games,
- can also lead to situations where only a single solution remains,
- but: the result can depend on the elimination order! (see example on next slide)
Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

<table>
<thead>
<tr>
<th>player 1</th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
</tr>
<tr>
<td></td>
<td>R</td>
</tr>
<tr>
<td>M</td>
<td>2,1</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
</tr>
</tbody>
</table>

- **Step 1**: eliminate row \( T \) (weakly dominated by row \( M \))
- **Step 2**: eliminate column \( L \) (weakly dominated by col. \( R \))

Different elimination order, different result, even different payoffs (1, 1 vs. 2, 1)!

4 Nash Equilibria

- Definitions and Examples
- Example: Sealed-Bid Auctions
- Iterative Elimination and Nash Equilibria
We write for a strategy profile $\langle a_i \rangle_{i \in N}$, then a strategy $a_i$ and its associated payoff $u_i(a_i)$.

Definition (Best response)
A strategy profile $\langle a_i \rangle_{i \in N}$ is a best response of player $i$ to the other players’ strategies $\langle a_{-i} \rangle_{i \in N \setminus \{i\}}$ if \[ u_i(a_i) = \max_{a'_i \in A_i} u_i(a'_i, a_{-i}). \]

Remark: In following examples, for non-Nash equilibria, only one possible profitable deviation is shown (even if there are more).

Possible answer:
- Strategy profiles where no player benefits from playing a different strategy.
- Equivalently: Strategy profiles where every player’s strategy is a best response to the other players’ strategies.

Such strategy profiles are called Nash equilibria, one of the most-used solution concepts in game theory.

Remark: There is an alternative definition of Nash equilibria (which we consider because it gives us a slightly different perspective on Nash equilibria).

Definition (Nash equilibrium)
A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that for every player $i \in N$, \[ u_i(a^*) \geq u_i(a_{-i}^*, a_i) \quad \text{for all } a_i \in A_i. \]

Possible answer:
- Strategy profiles where every player’s strategy is a best response to the other players’ strategies.
- Equivalently: Strategy profiles where no player benefits from playing a different strategy.

The three definitions of Nash equilibria are equivalent.

Proof.
Homework.
Nash Equilibria

Example (Nash Equilibria in the Prisoner’s Dilemma)

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
</tr>
<tr>
<td>player 1</td>
<td>3,3</td>
</tr>
<tr>
<td></td>
<td>4,0</td>
</tr>
</tbody>
</table>

- \((C, C)\): No Nash equilibrium (player 1: \(C \rightarrow D\))
- \((C, D)\): No Nash equilibrium (player 1: \(C \rightarrow D\))
- \((D, C)\): No Nash equilibrium (player 2: \(C \rightarrow D\))
- \((D, D)\): Nash equilibrium!

Example (Nash Equilibria in Hawk and Dove)

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>D</td>
</tr>
<tr>
<td>player 1</td>
<td>3,3</td>
</tr>
<tr>
<td></td>
<td>4,1</td>
</tr>
</tbody>
</table>

- \((D, D)\): No Nash equilibrium (player 1: \(D \rightarrow H\))
- \((D, H)\): Nash equilibrium!
- \((H, D)\): Nash equilibrium!
- \((H, H)\): No Nash equilibrium (player 1: \(H \rightarrow D\))

Example (Nash Equilibria in Matching Pennies)

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>H</td>
</tr>
<tr>
<td>player 1</td>
<td>1,−1</td>
</tr>
<tr>
<td></td>
<td>−1, 1</td>
</tr>
</tbody>
</table>

- \((H, H)\): No Nash equilibrium (player 2: \(H \rightarrow T\))
- \((H, T)\): No Nash equilibrium (player 1: \(H \rightarrow T\))
- \((T, H)\): No Nash equilibrium (player 1: \(T \rightarrow H\))
- \((T, T)\): No Nash equilibrium (player 2: \(T \rightarrow H\))

Example (Nash Equilibria in Bach or Stravinsky)

<table>
<thead>
<tr>
<th></th>
<th>Stravinsky enthusiast</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
</tr>
<tr>
<td>Bach enthusiast</td>
<td>2,1</td>
</tr>
<tr>
<td></td>
<td>0,0</td>
</tr>
</tbody>
</table>

- \((B, B)\): Nash equilibrium!
- \((B, S)\): No Nash equilibrium (player 1: \(B \rightarrow S\))
- \((S, B)\): No Nash equilibrium (player 2: \(S \rightarrow B\))
- \((S, S)\): Nash equilibrium!
Assume three bidders 1, 2, and 3, with valuations and bids

\[ v_1 = 100, \quad v_2 = 80, \quad v_3 = 53, \]
\[ b_1 = 90, \quad b_2 = 85, \quad b_3 = 45. \]

Observations:
- Bidder 1 wins, pays 90, gets utility
  \[ u_1(b) = v_1 - b_1 = 100 - 90 = 10. \]
- Bidders 2 and 3 pay nothing, get utility 0.
- (Bidder 2 over-bids.)
- Bidder 1 could still win, but pay less, by bidding \( b_1' = 85 \) instead. Then \( u_1(b_{-1}, b_1') = v_1 - b_1' = 100 - 85 = 15. \)

Question: How to avoid untruthful bidding and incentivize truthful revelation of private valuations?
Different answer to question about payments: Winner pays the second-highest bid.

Definition (Second-price sealed-bid auction)
- \( N = \{1, \ldots, n\} \) with \( v_1 > v_2 > \cdots > v_n > 0 \),
- \( A_i = \mathbb{R}^+_0 \) for all \( i \in N \),
- Bidder \( i \in N \) wins if \( b_i \) is maximal among all bids (+ possible tie-breaking by index), and
- \( u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - \max b_{-i} & \text{otherwise} \end{cases} \)
where \( b = (b_1, \ldots, b_n) \).
Example: Sealed-Bid Auctions

Example (Second-price sealed-bid auction)
Assume three bidders 1, 2, and 3, with valuations and bids

\[ v_1 = 100, \quad v_2 = 80, \quad v_3 = 53, \]
\[ b_1 = 90, \quad b_2 = 85, \quad b_3 = 45. \]

Observations:
- Bidder 1 wins, pays 85, gets utility
  \[ u_1(b) = v_1 - b_2 = 100 - 85 = 15. \]
- Bidders 2 and 3 pay nothing, get utility 0.
- Bidder 1 has no incentive to bid strategically and guess
  the other bidders' private valuations.

Proof (ctd.)
Ad (1) [regardless of what the other bidders do, \( b^*_i \) is always a best response]:

- Case I) bidder \( i \) wins:
  \( \text{bidder } i \text{ pays max } b_{-i} \leq v_i, \text{ gets } u_i(b_{-i}, b^*_i) \geq 0. \)
- Case I.a) bidder \( i \) decreases bid:
  this does not help, since he might still win and pay the
  same as before, or lose and get utility 0.
- Case I.b) bidder \( i \) increases bid:
  bidder \( i \) still wins and pays the same as before.

Proposition
In a second-price sealed-bid auction, bidding one's own
valuation, \( b^*_i = v_i \), is a weakly dominant strategy.

Proof.
We have to show that \( b^*_i \) weakly dominates every other
strategy \( b_i \) of player \( i \).
For that, it suffices to show that
- for all \( b_i \in A_i \), we have
  \( u_i(b_{-i}, b^*_i) \geq u_i(b_{-i}, b_i) \) for all \( b_{-i} \in A_{-i} \), and that
- for all \( b_i \in A_i \), we have
  \( u_i(b_{-i}, b^*_i) > u_i(b_{-i}, b_i) \) for at least one \( b_{-i} \in A_{-i} \).

Example: Sealed-Bid Auctions

Example (Second-price sealed-bid auction)
Assume three bidders 1, 2, and 3, with valuations and bids

\[ v_1 = 100, \quad v_2 = 80, \quad v_3 = 53, \]
\[ b_1 = 90, \quad b_2 = 85, \quad b_3 = 45. \]

Observations:
- Bidder 1 wins, pays 85, gets utility
  \[ u_1(b) = v_1 - b_2 = 100 - 85 = 15. \]
- Bidders 2 and 3 pay nothing, get utility 0.
- Bidder 1 has no incentive to bid strategically and guess
  the other bidders' private valuations.

Proof (ctd.)
Ad (1) (ctd.):

- Case II) bidder \( i \) loses:
  bidder \( i \) pays nothing, gets
  \[ u_i(b_{-i}, b^*_i) = 0. \]
- Case II.a) bidder \( i \) decreases bid:
  bidder \( i \) still loses and gets utility 0.
- Case II.b) bidder \( i \) increases bid:
  either bidder \( i \) still loses and gets utility 0, or becomes
  the winner and pays more than the object is worth to him,
  leading to a negative utility.
Motivation: We have seen two different solution concepts, surviving iterative elimination of (strictly) dominated strategies and Nash equilibria.

Obvious question: Is there any relationship between the two?

Answer: Yes, Nash equilibria refine the concept of iterative elimination of strictly dominated strategies. We will formalize this on the next slides.

Lemma (preservation of Nash equilibria)

Let $G$ and $G'$ be two strategic games where $G'$ is obtained from $G$ by elimination of one strictly dominated strategy. Then a strategy profile $a^*$ is a Nash equilibrium of $G$ if and only if it is Nash equilibrium of $G'$.

Proof.

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ and $G' = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$. Let $a'_i$ be the eliminated strategy. Then there is a strategy $a^*_i$ such that for all $a_{-i} \in A_{-i}$,$$u_i(a_{-i}, a^*_i) < u_i(a_{-i}, a'_i).$$
Iterative Elimination and Nash Equilibria

Proof (ctd.)

“⇒”: Let \( a^* \) be a Nash equilibrium of \( G \).
- Nash equilibrium strategies are not eliminated: For players \( j \neq i \), this is clear, because none of their strategies are eliminated.
- For player \( i \), action \( a_i^* \) is a best response to \( a_{-i}^* \), and in particular at least as good a response as \( a_i^* \):
  \[
  u_i(a_{-i}, a_i^*) \geq u_i(a_{-i}, a_i^*).
  \]
  With (1) \( u_i(a_{-i}, a_i^*) > u_i(a_{-i}, a_i^*) \), we get
  \[
  u_i(a_{-i}, a_i^*) > u_i(a_{-i}, a_i^*) \text{ and hence } a_i^* \neq a_i^*.
  \]
  Thus, the Nash equilibrium strategy \( a_i^* \) is not eliminated.

“⇐”: Let \( a^* \) be a Nash equilibrium of \( G' \).
- For player \( j \neq i \): \( a_j^* \) is a best response to \( a_{-j}^* \) in \( G \) as well, since the responses available to player \( j \) in \( G \) and \( G' \) are the same.
- For player \( i \): \( a_i^* \) is also a Nash equilibrium of \( G' \).

Proof (ctd.)

Corollary

If iterative elimination of strictly dominated strategies results in a unique strategy profile \( a^* \), then \( a^* \) is the unique Nash equilibrium of the original game.

Proof.

Assume that \( a^* \) is the unique remaining strategy profile. By definition, \( a^* \) must be a Nash equilibrium of the remaining game.

We can inductively apply the previous lemma (preservation of Nash equilibria) and see that \( a^* \) (and no other strategy profile) must have been a Nash equilibrium before the last elimination step, and before that step, \( \ldots \), and in the original game.
Playing it Safe (in Two-Player Games)

Motivation: What happens if both players try to “play it safe”?

Question: What does it even mean to “play it safe”?

Answer: Choose a strategy that guarantees the highest worst-case payoff.

Example

\[
\begin{array}{c|cc}
\text{L} & R & \text{B} \\
\text{T} & 2,1 & 2, -20 \\
\text{M} & 3, 0 & -10, 1 \\
\text{B} & -100,2 & 3, 3 \\
\end{array}
\]

Worst-case payoff for player 1:
- if playing \( T \): 2
- if playing \( M \): -10
- if playing \( B \): -100

\( \Rightarrow \) play \( T \).

Worst-case payoff for player 2:
- if playing \( L \): 0
- if playing \( R \): -20

\( \Rightarrow \) play \( L \).

However: Unlike \((B, R)\), the profile \((T, L)\) is not a Nash equilibrium.
Preliminaries and Examples

Solution Concepts and Notation

Dominated Strategies

Nash Equilibria

Zero-Sum Games

Summary

Playing it Safe (in Two-Player Games)

Example

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>2, 1</td>
</tr>
<tr>
<td>T</td>
<td>2, -20</td>
</tr>
<tr>
<td>M</td>
<td>3, 0</td>
</tr>
<tr>
<td>B</td>
<td>-100, 2</td>
</tr>
</tbody>
</table>

Worst-case payoff for player 1:
- if playing T: 2
- if playing M: -10
- if playing B: -100

Worst-case payoff for player 2:
- if playing L: 0
- if playing R: -20

⇝ play T.

However: Unlike (B, R), the profile (T, L) is not a Nash equilibrium.

Zero-Sum Games

Definition (Zero-sum game)

A zero-sum game is a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and

$$u_1(a) = -u_2(a)$$

for all $a \in A$.

Example (Matching Pennies as a zero-sum game)

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1, -1</td>
</tr>
<tr>
<td>T</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>

Maximimizers

Definition (Maximinimizer)

Let $G = \langle \{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a zero-sum game.

An action $x^* \in A_1$ is called maximinimizer for player 1 in $G$ if

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y)$$

for all $x \in A_1$,

and $y^* \in A_2$ is called maximinimizer for player 2 in $G$ if

$$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y)$$

for all $y \in A_2$.
Maximinimizers

Example (Zero-sum game with three actions each)

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>player 1</td>
<td>T</td>
<td>8</td>
<td>−8</td>
</tr>
<tr>
<td></td>
<td>M</td>
<td>2</td>
<td>−2</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>−6</td>
<td>6</td>
</tr>
<tr>
<td>player 2</td>
<td>L</td>
<td>3</td>
<td>−3</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>1</td>
<td>−1</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>8</td>
<td>−6</td>
</tr>
</tbody>
</table>

Guaranteed worst-case payoffs:

- T: −6, M: −1, B: −6 ⇒ maximinimizer M
- L: −8, C: −4, R: −8 ⇒ maximinimizer C

⇒ pair of maximinimizers (M, C) with payoffs (−1, 1)

(not a Nash equilibrium; this game has no Nash equilibrium.)

Maximinimizers

Lemma

Let \( G = ( \{1, 2\}, (A_i)_{i \in \mathbb{N}}, (u_i)_{i \in \mathbb{N}} ) \) be a zero-sum game. Then

\[
\max_y \min_x u_2(x, y) = - \min_y \max_x u_1(x, y). \tag{2}
\]

Proof.

For any real-valued function \( f \), we have

\[
\min_z -f(z) = - \max_z f(z). \tag{3}
\]

Maximinimizers

Example (Maximinimization vs. minimaximization)

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>player 1</td>
<td>T</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>M</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>−2</td>
</tr>
<tr>
<td>player 2</td>
<td>L</td>
<td>−1</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>−4</td>
</tr>
</tbody>
</table>

Worst-case payoffs (player 2):

- L: −1, R: −2
- L: −1, R: −2
- Maximize: −1

Best-case payoffs (player 1):

- L: −1, R: −2
- Maximize: +1

Observation: Results identical up to different sign.

Proof (ctd.)

Thus, for all \( y \in A_2 \),

\[
- \min_y \max_x u_1(x, y) \geq \min_y \max_x u_1(x, y) \Rightarrow \max_y -u_1(x, y) \geq \min_y -u_1(x, y). \tag{3}
\]

\[
\max_y \min_x u_2(x, y). \tag{3}
\]
Now, we are ready to prove our main theorem about zero-sum games and Nash equilibria.

In zero-sum games:
1. Every Nash equilibrium is a pair of maximinimizers.
2. All Nash equilibria have the same payoffs.
3. If there is at least one Nash equilibrium, then every pair of maximinimizers is a Nash equilibrium.

\[\text{SS 2019 B. Nebel, R. Mattmüller – Game Theory 70 / 84}\]

**Theorem (Maximinimizer theorem)**

Let \( G = \langle \{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \) be a zero-sum game. Then:

1. If \((x^*, y^*)\) is a Nash equilibrium of \( G \), then \(x^*\) and \(y^*\) are maximinimizers for player 1 and player 2, respectively.
2. If \((x^*, y^*)\) is a Nash equilibrium of \( G \), then

\[
\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y) = u_1(x^*, y^*),
\]

and \(x^*\) and \(y^*\) maximinimizers of player 1 and player 2 respectively, then \((x^*, y^*)\) is a Nash equilibrium.

\[\text{SS 2019 B. Nebel, R. Mattmüller – Game Theory 71 / 84}\]

**Proof.**

1. Let \((x^*, y^*)\) be a Nash equilibrium. Then

\[u_2(x^*, y^*) \geq u_2(x^*, y)\quad \text{for all } y \in A_2.\]

With \(u_1 = -u_2\), this implies

\[u_1(x^*, y^*) \leq u_1(x^*, y)\quad \text{for all } y \in A_2.\]

Thus

\[u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y) \leq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \tag{4}\]

\[\text{SS 2019 B. Nebel, R. Mattmüller – Game Theory 72 / 84}\]

**Proof (ctd.)**

1. (ctd.) Furthermore, since \((x^*, y^*)\) is a Nash equilibrium, also

\[u_1(x^*, y^*) \geq u_1(x, y^*)\quad \text{for all } x \in A_1.\]

Hence

\[u_1(x^*, y^*) \geq \max_{x \in A_1} u_1(x, y^*).\]

This implies

\[u_1(x^*, y^*) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \tag{5}\]

\[\text{SS 2019 B. Nebel, R. Mattmüller – Game Theory 73 / 84}\]
Nash Equilibria in Zero-Sum Games

Proof (ctd.)

(c) Let \( x^* \) and \( y^* \) be maximinimizers for player 1 and 2, respectively, and assume that
\[
\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y) =: v^*.
\]
(8)
With Equation (2) from the previous lemma, we get
\[
\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -v^*.
\]
(9)
With \( x^* \) and \( y^* \) being maximinimizers, (8) and (9) imply
\[
u_1(x^*,y) \geq v^* \quad \text{for all } y \in A_2, \quad \text{and} \quad u_2(x,y^*) \geq -v^* \quad \text{for all } x \in A_1.
\]
(10)
(11)
In particular, it follows that all Nash equilibria share the same payoff profile.
Proof (ctd.)

Similarly, we can plug (12) into the right-hand side of (11) and obtain

$$u_2(x, y') \geq -u_1(x^*, y^*)$$ for all $x \in A_1$.

Again using $u_1 = -u_2$, this is equivalent to

$$u_1(x, y') \leq u_1(x^*, y^*)$$ for all $x \in A_1$.

In other words, $x^*$ is also a best response to $y^*$.

Hence, $(x^*, y^*)$ is a Nash equilibrium.

\[\square\]

Corollary

Let $G = \langle \{1, 2\}, (A_i)_{i \in \mathbb{N}}, (u_i)_{i \in \mathbb{N}} \rangle$ be a zero-sum game, and let $(x_1^*, y_1^*)$ and $(x_2^*, y_2^*)$ be two Nash equilibria of $G$.

Then $(x_1^*, y_2^*)$ and $(x_2^*, y_1^*)$ are also Nash equilibria of $G$.

In other words: Nash equilibria of zero-sum games can be arbitrarily recombined.
Strategic games are one-shot games of finitely many players with given action sets and payoff functions. Players have perfect information.

Solution concepts: survival of iterative elimination of strictly dominated strategies, Nash equilibria.

Relation between solution concepts: Nash equilibria always survive iterative elimination of strictly dominated strategies.

In zero-sum games, one player’s gain is the other player’s loss. Thus, playing it safe is rational. Relevant concept: maximinimizers.

Relation to Nash equilibria: In zero-sum games, Nash equilibria are pairs of maximinimizers, and, if at least one Nash equilibrium exists, pairs of maximinimizers are also Nash equilibria.