Department of Computer Science
Research Group Foundations of Artificial Intelligence
Prof. Dr. B. Nebel

Exam:	**Game Theory**
Date and time:	August 28, 2018, 10:00 a.m.
Duration:	90 minutes
Room:	HS 101-00-036
Permitted exam aids:	Indelible pen (e. g. ball pen, *no* pencil!), nothing else.
Examiner:	Prof. Dr. Bernhard Nebel

Last name:	*LASTNAME*
First name:	*FIRSTNAME*
Matriculation number:	*MATRICULATIONNUMBER*
Signature:	....

Notes:

- Please fill out this form.
- Please write only on one side of your paper sheets.
- Please use a new paper sheet for each question.
- Please turn off your mobile phone.
- You may answer the questions in English or in German.

Withdrawing from an examination:
In case of illness, you must supply proof of your illness by submitting a medical report to the Examinations Office. Please note that the medical examination must be done at the latest on the same day of the missed exam.

Cheating/disturbing in examinations:
A student who disrupts the orderly proceedings of an examination will be excluded from the remainder of the exam by the respective examiners or invigilators. In such a case, the written exam of the student in question will be graded as “nicht bestanden” (5.0, fail) on the grounds of cheating. In severe cases, the Board of Examiners will exclude the student from further examinations.

Question	**Score**	**Reached score**	**Comments**	**Initials**
1	**18**			
2	**10**			
3	**12**			
4	**18**			
5	**14**			
6	**18**			
Sum	**90**			

Grade: ....

Date of the review of the exam: ....

Signature of the examiner: ....

Question 1 (6+6+6 points) Strategic Games

(a) Consider the following strategic game:

		Player 2			
		X	Y	Z	U
Player 1	A	$1,1$	$4,0$	$2,6$	$3,4$
	B	$5,6$	$1,3$	$0,3$	$5,2$
	C	$6,2$	$5,1$	$1,1$	$4,0$
	D	$0,4$	$8,3$	$1,5$	$2,6$

Apply the method of iterative elimination of strictly dominated strategies. Highlight all pure-strategy Nash equilibria in the game matrix.

Solution:

- X dom Y (1p)
- A dom D (1p)
- Z dom U (1p)
- C dom B (1p)
- NE: (C,X), (A,Z) (jew. 1p)

(b) Consider the following strategic game:

		Player 2	
		X	Y
Player 1	A	$x,4$	$0,0$
	B	$0,0$	$2,2$

Let α be a mixed-strategy Nash equilibrium with $0 < \alpha_1(A) < 1$ and $0 < \alpha_2(X) < 1$ and let $x \in \mathbb{R}^+$. How do $\alpha_1(A)$ and $\alpha_2(X)$ change as x is decreased? Justify your answer.

Solution:
Both answers can be justified by simple equations. In that case, no textual explanation is required.

- α_1 does not change (1.5p)
- because player 2's values do not change (1.5p).
- $\alpha_2(X)$ increases (1.5p)
- because player 1 has to be indifferent between playing A and B against α_2 (1.5).

(c) Consider the following strategic game:

		Player 2		
		X	Y	Z
Player 1	A	$1,1$	$2,1$	$0,2$
	B	$1,1$	$0,1$	$2,2$
	C	$2,2$	$1,0$	$1,1$

Determine all Nash equilibria (pure or mixed). Explain how you arrived at your solution.

(*Hint:* You do not need to consider all pairs of support-sets for the computation of the mixed-strategy Nash equilibria. It may be useful to simplify the game first.)

Solution:
Strategies A and Y can be eliminated since they are strictly dominated.

- pure NE: (B, Z) (1p)
- pure NE: (C, X) (1p)
- mixed NE: $(\alpha_1, \alpha_2) = ((0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2}))$ (2p, for α_1, 2p for α_2)

(additional room for answer to Question 1)

Question 2 (8+2 points) CORRELATED EQUILIBRIA

(a) Consider the traffic game defined by the following payoff matrix:

		Player 2: *Go*	Player 2: *Wait*
Player 1	*Go*	$-10, -10$	$1, 0$
	Wait	$0, 1$	$-1, -1$

The Nash equilibrium payoff profiles are $(0, 1)$ and $(1, 0)$ (pure) and $(-\frac{5}{6}, -\frac{5}{6})$ (mixed). Construct a correlated equilibrium that yields a payoff profile such that both players have a higher payoff than in the mixed payoff profile. To that end, specify the probability space (Ω, π), the information partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, and strategies σ_1 and σ_2 on them, and show that this forms indeed a correlated equilibrium.

Solution:
Let $\Omega = \{red, green\}$ and $\pi(red) = \pi(green) = \frac{1}{2}$. Let $\mathcal{P}_1 = \mathcal{P}_2 = \{\{red\}, \{green\}\}$. Define the strategies as follows:

$$\sigma_1(red) = Go,\ \sigma_1(green) = Wait$$
$$\sigma_2(red) = Wait,\ \sigma_2(green) = Go$$

Both players play optimally and get a payoff profile of $(\frac{1}{2}, \frac{1}{2})$.

(b) Briefly describe the connection of correlated equilibria to Nash equilibria.

Solution:
Corr. equilibria are a generalization of NE. For each NE/MSNE we can construct a corr. equilibrium.

(additional room for answer to Question 2)

Question 3 (4+4+2+2 points) COMPLEXITY

(a) Is the problem of computing mixed-strategy Nash equilibria for finite zero-sum games exponential or not? Justify your answer.

(b) Order the following complexity classes with respect to class inclusion from smallest to largest: TFNP, FP, FNP, PPAD.

(c) The search problem NASH consists of finding a mixed-strategy Nash equilibrium for a given finite two-player strategic game. Name the complexity class for which NASH is complete. Name the prototypical complete problem for this complexity class used to prove this result.

(d) The search problem 2ND-NASH consists of finding a second, different mixed-strategy Nash equilibrium for a given finite two-player strategic game and a first mixed-strategy Nash equilibrium for it. Specify the computational complexity of 2ND-NASH.

(additional room for answer to Question 3)

Question 4 (8+2+8 points) REPEATED GAMES

Consider the infinitely repeated prisoner's dilemma. The payoff matrix of the stage game is given below.

		Player 2	
		C	D
Player 1	C	$3,3$	$0,10$
	D	$10,0$	$1,1$

(a) Under the discounting preference criterium, for which discount factor $0 < \delta < 1$ is (GRIM, GRIM) a Nash equilibrium? Justify your answer.

(*Hint:* The GRIM strategy starts with playing C. After any play of D it plays D forever.)

Solution:

W.l.o.g. assume that player 1 deviates in the first round. After the first deviation player 1 can never get more than 1 utility, since player 2 will always defect.

$$\begin{aligned} v_1(O(s,g)) &= 10 + 1\delta + 1\delta^2 + 1\delta^3 + \dots \\ &= 10 + \sum_{i=0}^{N} \delta^i - 1 \\ &= 9 + \frac{1}{1-\delta} \\ v_1(O(g,g)) &= \frac{3}{1-\delta} \end{aligned}$$

A deviation is not profitable if

$$\begin{aligned} 9 + \frac{1}{1-\delta} &\leq \frac{3}{1-\delta} \\ \Leftrightarrow \delta &\geq \frac{7}{9} \end{aligned}$$

(GRIM, GRIM) is a NE for $\delta \geq \frac{7}{9}$.

(i.e. if the players care about tomorrow at least $\frac{7}{9}$ as much as today.)

(b) What is player 1's minimax payoff?

Solution:

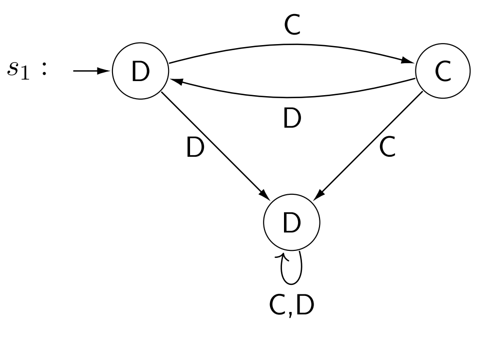
$v_1 = \min_{a_2 \in A_2} \max_{a_1 \in A_1} u_1(a_1, a_2) = 1$

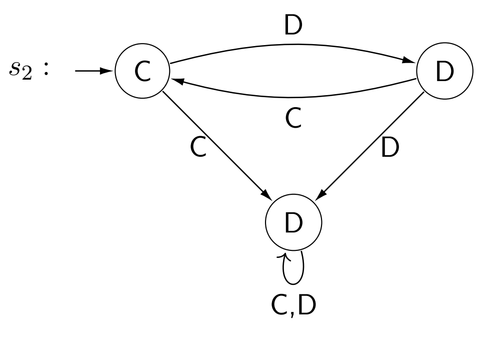
(c) Consider the following payoff profiles under the limit-of-means preference criterium. For each payoff profile, either construct two automata that form a Nash equilibrium or argue that no Nash equilibrium with the given payoffs exists.

- $(5,5)$

  Solution:

  (s_1, s_2) is a NE with $v(O(s_1, s_2)) = (5,5)$, where



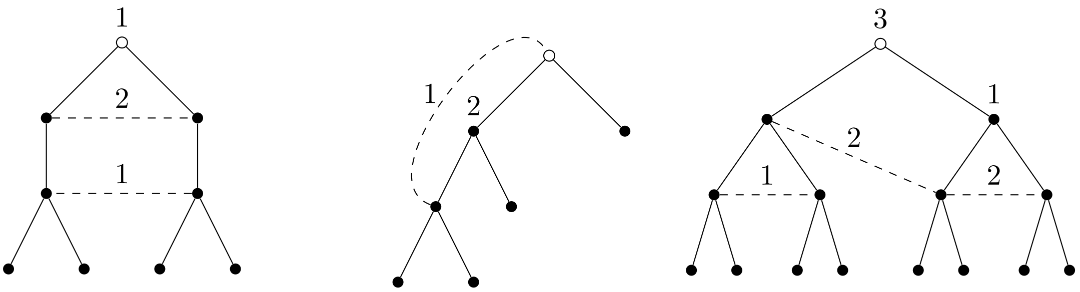


- $(6, 6)$
  Solution:
  enforceable but not feasible
- $(3, 0)$
  Solution:
  not enforceable and not feasible

(additional room for answer to Question 4)

Question 5 (6+2+6 points) Imperfect Information Games

Consider the following three extensive games with imperfect information:



(i) (ii) (iii)

(a) For each game (i), (ii), (iii) state whether it is a game of perfect recall. Justify your answer.

Solution:

To be a game of perfect recall, all players must (1) always remember what they have learned before and (2) which actions they have performed.

Def.: An ext. game has perfect recall if for each player i, we have $X_i(h) = X_i(h')$ whenever the histories h and h' are in the same information set of player i.

(i) no (1p): player 1 cannot remember his first move (1p)

(ii) no (1p): player 1 forgets that he made a move (1p)

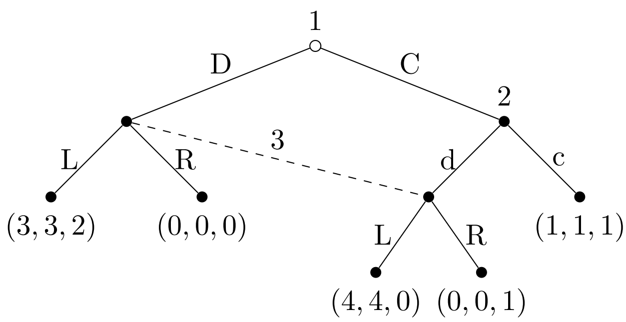
(iii) yes (1p): player 1 and 2 do not know whether they go first but have perfect recall (1p)

(b) In as few words as possible, explain the difference between a mixed strategy and a behavioral strategy.

Solution:

Mixed = probability distribution over the set of a players pure strategies. (1p)
Behavioral = a collection of independent prob. distributions. For each history $h \in I_i \in \mathcal{I}_i$ and action $a \in A(h)$ define the probability that a is played at h. (1p)

(c) Consider the following strategic game with imperfect information:



Consider the Nash equilibrium β, with $\beta_1(\langle\rangle)(D) = 1, \beta_2(\langle C\rangle)(c) = \frac{1}{3}$, and $\beta_3(\{\langle D\rangle, \langle C, d\rangle\})(L) = 1$. Does a belief μ exist, such that (β, μ) is a sequential equilibrium? Justify your answer.

Solution:

- (β, μ) is not a sequential equilibrium (for any μ) (1p) because

- sequential rationality is violated at players 2's information partition $I_2 = \{\langle C \rangle\}$. (2p)

Consider β_2' with $\beta_2'(\langle C \rangle)(d) = 1$:

$$\begin{aligned}
U_2(\beta, \mu | I_2) &= \mu(I_2)(\langle C \rangle) \cdot \beta_2(I_2)(d) \cdot 4 + \mu(I_2)(\langle C \rangle) \cdot (1 - \beta_2(I_2)(d)) \\
&= \beta_2(I_2)(d) \cdot 4 + (1 - \beta_2(I_2)(d)) \cdot 1 \\
&= \frac{2}{3} \cdot 4 + \frac{1}{3} \cdot 1 \\
&= 3 \\
&< 4 = 1 \cdot 4 = U_2(\beta_2', \beta_{-2}, \mu | I_2)
\end{aligned}$$

(3p) for showing how seq. rationality is violated.

(additional room for answer to Question 5)

Question 6 (4+8+6 points) VICKREY-CLARKE-GROVES MECHANISM

(a) VCG mechanisms are (1) *social welfare maximizing* and (2) *incentive compatible.* Briefly describe what these two properties mean.

(b) In a *k-item auction*, k identical items are to be sold. Each bidder $i = 1, \dots, n$ can get at most one of the items and has a privately known valuation w_i for the item. For simplicity, assume that $w_1 > w_2 > \dots > w_n$. The set of alternatives $A = N_k$ consists of all k-ary subsets of players. Each alternative represents the players who will receive an item. Formalize the k-item auction as a VCG mechanism $\mathcal{M} = \langle f, (p_i)_{i\in N}\rangle$ that uses *Clarke pivot functions.*

Solution:
The players valuations over the alternatives $a \in A$ are

$$v_i(a) = \begin{cases} w_i, & \text{if } i \in a. \ (2p) \\ 0, & \text{otherwise. } (1p) \end{cases}$$

$$f(v_1, \dots, v_n) = \{i \in N | 1 \le i \le k\} \ (2p)$$

$$p_i(a) = \begin{cases} w_{k+1}, & \text{if } i \in a. \ (1.5p) \\ 0, & \text{otherwise. } (1.5p) \end{cases}$$

Alternative:

$$p_i(a) = \max_{b\in A} \sum_{j\neq i} v_j(b) - \sum_{j\neq i} v_j(a)$$

(1.5p) for correct Clarke Pivot function, (1.5p) for subtraction of social welfare max.

$$p_1(a) = (w_2 + w_3 + \dots + w_{k+1}) - (w_2 + w_3 + \dots + w_k) = w_{k+1}$$
$$\vdots$$
$$p_{k+1}(a) = (w_1 + w_2 + \dots + w_k) - (w_1 + w_2 + \dots + w_k) = 0$$
$$\vdots$$

This is also ok. (1.5p) for w_{k+1}'s and (1.5p) for 0's.

(c) Consider the mechanism $\mathcal{M}' = \langle f', (p'_i)_{i\in N}\rangle$ implementing a k-item auction, with social choice function

$$f'(v_1, \dots, v_n) = \{i \in N \mid 1 \le i \le k\},$$

and payment functions

$$p'_i(a) = \begin{cases} w_{i+1}, & \text{if } i \in a, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for all } a \in A.$$

Here, the i-th highest bidding winner has to pay the $(i+1)$-st highest bid, i. e., the highest bidding player pays the second highest bid, the second highest bidder pays the third highest bid, and so on. Non-winning players pay nothing.

Construct a counterexample that proves that $\mathcal{M}'$ is *not* incentive compatible.

(*Hint:* There is a counterexample with only three bidders.)

Solution:

- correct players' privately known valuations w_i (2p)
  e.g.: $w_1 = 10, w_2 = 8, w_3 = 0$
- correct beneficial deviation for one player (2p)
  e.g.: $w_1' = 7, w_2' = w_2, w_3' = w_3$
- showing that the deviation is beneficial (2p)
  e.g.: $p_1(w_1, w_2, w_3) = 8 > 0 = p_1(w_1', w_2, w_3)$
  therefore: $u_1(w) = 2 < 10 = u_1(w')$.

(additional room for answer to Question 6)

FIRSTNAME LASTNAME