**Exercise 3.1 (S5: Axioms and frame properties I, 6 points)**

A Kripke frame $\mathcal{F} = \langle S, R \rangle$ is defined exactly like a Kripke model $\langle S, R, V \rangle$, but without the valuation $V$. The set of all models over $\langle S, R \rangle$ is the set of all models $\langle S, R, V \rangle$ where $V$ is any propositional valuation. A formula is valid in a frame $\mathcal{F}$, if it is valid in all models over $\mathcal{F}$. It is valid in a class of frames, if it is valid in each frame in that class. We say that an axiom defines a class of frames if the axiom is valid exactly in this class of frames. Show that

(a) the axiom $T$ defines the class of reflexive frames,

(b) the axiom $4$ defines the class of transitive frames,

(c) the axiom $5$ defines the class of Euclidean frames.

Note: You might be able to re-use parts of your solutions for Exercise 2.3.

**Exercise 3.2 ($n$-bisimulation, 4 points)**

Let two models $\mathcal{M} = \langle S, R, V \rangle$ and $\mathcal{M}' = \langle S', R', V' \rangle$ be given. For any natural number $n$, we define two states $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ to be $n$-bisimilar, writing $(\mathcal{M}, s) \sim_n (\mathcal{M}', s')$, iff

$(\text{atoms})$ $s \in V(p)$ iff $s' \in V'(p)$ for all $p \in P$,

$(\text{forth})$ $n = 0$ or (if $n > 0$) for all $a \in A$ and $t \in S$ such that $(s, t) \in R_a$, there is also a $t' \in S'$ such that $(s', t') \in R'_a$ and $(\mathcal{M}, t) \equiv_{n-1} (\mathcal{M}', t')$, and

$(\text{back})$ $n = 0$ or (if $n > 0$) for all $a \in A$ and $t' \in S'$ such that $(s', t') \in R'_a$, there is also a $t \in S$ such that $(s, t) \in R$ and $(\mathcal{M}, t) \equiv_{n-1} (\mathcal{M}', t')$.

Furthermore, we define the modal depth of $L_K$-formulas as

\[
\begin{align*}
\text{depth}(p) &= 0 \text{ if } p \text{ is an atomic proposition} \\
\text{depth}(\neg \phi) &= \text{depth}(\phi) \\
\text{depth}(\phi \land \psi) &= \max\{\text{depth}(\phi), \text{depth}(\psi)\} \\
\text{depth}(K_a \phi) &= 1 + \text{depth}(\phi).
\end{align*}
\]

We say that the states $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ are epistemically equivalent up to depth $n \in \mathbb{N}$ and write $(\mathcal{M}, s) \equiv^n_{L_K} (\mathcal{M}', s')$ if and only if $\mathcal{M}, s \models \phi$ iff $\mathcal{M}', s' \models \phi$ for all formulas $\phi \in L_K$ with $\text{depth}(\phi) \leq n$. Show that $(\mathcal{M}, s) \equiv^n_{L_K} (\mathcal{M}', s')$ if and only if $(\mathcal{M}, s) \equiv_n (\mathcal{M}', s')$.

**Exercise 3.3 (S5: Deriving theorems, 1+1 points)**

Derive the following S5 theorems. Recall that a derivation is a finite sequence of formulas, such that each formula is either an instance of one of the axioms, an instance of a propositional tautology, or the result of the application of one of the rules (necessitation, modus ponens) on previous formulas.

(a) $K_a(p \rightarrow p)$

(b) $C_B p \leftrightarrow C_B C_B p$