Public Announcements

So far: Only static knowledge
(Or, where knowledge changed over time, we discussed this change only intuitively, not formally.)

Now: How to model change of knowledge over time?

Note: Knowledge may change in different ways, e.g., via public or private announcements, by sensing, or by ontic (world-changing) actions that affect knowledge along the way.

This chapter: Only public announcements.

Announcement = public and truthful announcement

Example
I announce the fact: “The sun is shining”.
This announcement makes the fact common knowledge.

This holds for all public announcements of true facts about the world.
It does not generally hold for all public announcements of true statements about knowledge.
Public Announcements

Example (Unsuccessful update)
I announce: “p is true, but Bob does not know it” (p ∧ ¬K_b p).
As Bob hears my announcement, he now knows p, and the announced formula p ∧ ¬K_b p becomes false!

Intuition: How should epistemic models look like before and after?

Before:  
\[ \begin{array}{c|c|c} \ \ \ | \ p & b & \neg p \ \\ \hline a & & \ \\ \end{array} \]

After: Only those states survive where the announced formula is true.  
\[ \begin{array}{c|c|c} \ \ \ | \ p & b & \neg p \ \\ \hline a & & \ \\ \end{array} \]

Public Announcements

Example (ctd.)
Anne, Bill and Cath have drawn one card from a stack of three cards, 0, 1, 2. Anne has drawn 0, Bill has drawn 1 and Cath the 2.

Anne says: “I do not have card 1". (¬1_a)
Bill states: “I don’t know Anne’s card". (¬(K_b 0_a ∨ K_b 1_a ∨ K_b 2_a))
Anne says: “I know Bill’s card”. (K_a 0_b ∨ K_a 1_b ∨ K_a 2_b)
Anne says: “I have 0, Bill has 1, Cath has 2.” (0_a ∧ 1_b ∧ 2_c)

Announcement Syntax

Example
Anne, Bill and Cath have drawn one card from a stack of three cards, 0, 1, 2. Anne has drawn a 0, Bill has drawn a 1 and Cath the 2.

Notation: We write 0_a for the fact that Anne has card 0, etc. In order to describe states, we write three digits for Anne’s, Bill’s, and Cath’s card, e.g., 012 to describe the actual card distribution.

Hexa:
\[ \begin{array}{ccc} 012 & 021 & 210 \\ 021 & 120 & 201 \\ 102 & 212 & 012 \end{array} \]

Hexa, 012 \models K_a ¬(K_b 0_a ∨ K_b 1_a ∨ K_b 2_a),
Hexa, 012 \models K_a ¬1_a.
Definition (Languages $\mathcal{L}_{K[]}$ and $\mathcal{L}_{K\mathcal{C}}$)

Let $P$ be a countable set of atomic propositions and $A$ be a finite set of agent symbols. Then the language $\mathcal{L}_{K\mathcal{C}}$ is defined by the following BNF:

$$\phi ::= p \mid \neg \phi \mid (\phi \land \phi) \mid K_a \phi \mid C_B \phi \mid \langle \phi \rangle \psi,$$

where $p \in P$, $a \in A$, and $B \subseteq A$.

The language $\mathcal{L}_{K[]}$ is the same without the $C_B$ clause.

$\langle \phi \rangle \psi$ reads “after a truthful announcement of $\phi$, it holds that $\psi$”. $\langle \phi \rangle \psi$ is the dual of $[\phi] \psi$: “after some truthful announcement of $\phi$, it holds that $\psi$”.

Example

In (Hexa, 012), after Anne announces $\neg 1_a$, Cath knows that $0_a$:

Hexa.012 $\models [\neg 1_a]K_c 0_a$

After Bill’s announcement that he does not know Anne’s card, Anne knows Bill’s card:

Hexa.012 $\models [\neg (K_b 0_a \lor K_b 1_a \lor K_b 2_a)]K_a 1_b$

or: Hexa’.012 $\models [\neg (K_b 0_a \lor K_b 1_a \lor K_b 2_a)]K_a 1_b$

Announcement Semantics

Recall that, for models $\mathcal{M}$ with domain $S$ and formulas $\phi$, we write $[\phi]_{\mathcal{M}} = \{ s \in S \mid \mathcal{M}, s \models \phi \}$.

Definition

Let $\mathcal{M} = (S, R, V)$ be an epistemic model and $\phi$ a formula. Then $\mathcal{M} |_{\phi} = (S', R', V')$ with

- $S' = [\phi]_{\mathcal{M}}$,
- $R'_a = R_a \cap (S' \times S')$ for all $a \in A$, and
- $V'(p) = V(p) \cap S'$. 
Public Announcements
Semantics

Definition
The truth of an $\mathcal{L}_\mathcal{K}$ (or $\mathcal{L}_{\mathcal{KC}}$) formula $\phi$ in an epistemic state $(\mathcal{M}, s)$, symbolically $\mathcal{M}, s \models \phi$, is defined as for $\mathcal{L}_\mathcal{K}$ (or $\mathcal{L}_{\mathcal{KC}}$), with an additional clause for public announcements:

$\mathcal{M}, s \models [\phi] \psi$ iff $(\mathcal{M}, s \models \phi \text{ implies } \mathcal{M}|_{\phi}, s \models \psi)$.

Note: $[\phi] \psi$ is satisfied in $s$ if $\phi$ is not satisfied in $s$.

The dual $\langle \phi \rangle \psi = \neg[\phi] \neg\psi$ has the truth condition $\mathcal{M}, s \models \phi$ and $\mathcal{M}|_{\phi}, s \models \psi$.

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Announcements and Revelations

Question: Who actually makes the announcement?
- One of the agents?
- An omniscient external entity?

Observation: This makes a difference!
- If agent $a$ announces $\phi$, she must know $\phi$, and could also announce $K_a \phi$. This can make a difference!
- If the announcement comes from the outside, it is just $[\phi]$. This is also called a revelation.

Principles of Public Announcement Logics

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Motivation: In this section, we will prove some valid formulas of the language $L_K$ that will ultimately allow us to reduce $L_K$ to $L$ and get rid of announcement modalities.

Proposition (Functionality)

It is valid that $⟨φ⟩ψ → [φ]ψ$.

Proof.

Let $M, s$ be arbitrary. Assume that $M, s \models ⟨φ⟩ψ$. This is true if and only if $M, s \models φ$ and $M|φ, s \models ψ$. This implies that $M, s \models φ$ implies $M|φ, s \models ψ$, i.e., $M, s \models [φ]ψ$.

Question: What about the opposite direction? Is $[φ]ψ → ⟨φ⟩ψ$ also valid?

Proposition

$[φ]ψ → ⟨φ⟩ψ$ is not valid.

Proof.

Counterexample: model $M$ with a single state $s$ where atom $p$ is false. Then $M, s \models [p]p$, but $M, s \not\models ⟨p⟩p$.

Proposition (Partiality)

$⟨φ⟩⊤$ is not valid.

Proof.

In any epistemic state $(M, s)$ with $M, s \not\models φ$, we have $M, s \not\models ⟨φ⟩⊤$.
Introduction

Principles of Public Announcement Logics

Proposition (Negation)

\[
[\varphi] \neg \psi \leftrightarrow (\varphi \rightarrow \neg [\varphi] \psi)
\]
is valid.

Proof.

Omitted. Note that the biimplication can be equivalently written as

\[
[\varphi] \neg \psi \leftrightarrow (\neg \varphi \vee \langle \varphi \rangle \neg \psi).
\]
Let us now study how knowledge changes with announcements.

We find that $[\varphi]K_a\psi$ is not equivalent to $K_a[\varphi]\psi$.

Counterexample: Hexa.012 $\models [1_a]K_0\varphi$, but Hexa.012 $\not\models K_0[1_a]\varphi$.

Proposition (Knowledge)
$[\varphi]K_a\psi$ is equivalent to $\varphi \rightarrow K_a[\varphi]\psi$.

Proof.
\[ M, s \models \varphi \rightarrow K_a[\varphi]\psi \quad \text{iff} \quad M, s \models \varphi \implies M|_{\varphi}, t \models \psi \]
for all $t$ such that $(s, t) \in R_a$

\[ M, s \models \varphi \implies M|_{\varphi}, t \models \psi \quad \text{for all } t \in S \]

\[ M, s \models \varphi \implies (s, t) \in R_a \implies M|_{\varphi}, t \models \psi \]
for all $t \in [\varphi]$.

\[ M, s \models \varphi \implies (M|_{\varphi}, s \models K_a\psi) \]
\[ M, s \models [\varphi]K_a\psi. \]

Note: Using this proposition, one can reduce any $\mathcal{L}_{K[\square]}$ formula to an $\mathcal{L}_K$ formula. This means that both logics are equally expressive, and that we can use $\mathcal{L}_K$ theorem provers or model checkers for $\mathcal{L}_{K[\square]}$ as well.

Proposition (Reduction)

All of the following schemas are valid:

1. $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$ for all $p \in P$
2. $[\varphi](\psi \land \chi) \leftrightarrow ([\varphi]\psi \land [\varphi]\chi)$
3. $[\varphi](\psi \rightarrow \chi) \leftrightarrow ([\varphi]\psi \rightarrow [\varphi]\chi)$
4. $[\varphi] \neg \psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
5. $[\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[\varphi]\psi)$
6. $[\varphi][\psi]\chi \leftrightarrow [\varphi \land [\varphi]\psi]\chi$

Proof.
We already showed (4), (5), and (6). The others are an easy homework exercise.
Announcements and Common Knowledge

Counterexample:

Before announcement of $p$:

\[ s : p, q \]

After announcement of $p$:

\[ s' : \neg p, q \]

\[ s'' : p, \neg q \]

\[ \mathcal{M}, s \models [p]C_{ab}q \]

\[ \mathcal{M}, s \not\models p \rightarrow C_{ab}[p]q \]

Announcements and Common Knowledge

**Question:** Can we also systematically eliminate announcement modalities as shown above in the presence of the common knowledge modality?

Recall:

\[ [\varphi]K_{a}\psi \leftrightarrow (\varphi \rightarrow K_{a}[\varphi]\psi) \] is valid.

Attempted generalization to common knowledge:

\[ [\varphi]C_{B}\psi \leftrightarrow (\varphi \rightarrow C_{B}[\varphi]\psi). \]

Problem: This is invalid!

Announcements and Common Knowledge

**So, how to relate announcements and common knowledge?**

**Proposition (Announcements and common knowledge)**

If $\chi \rightarrow [\varphi]\psi$ and $(\chi \land \varphi) \rightarrow E_{B}\chi$ are valid, then $\chi \rightarrow [\varphi]C_{B}\psi$ is valid.

**Proof.**

Let $\mathcal{M}, s$ be arbitrary and suppose that $\mathcal{M}, s \models \chi$. We want to show that $\mathcal{M}, s \models [\varphi]C_{B}\psi$. Suppose $\mathcal{M}, s \models \varphi$, and let $t$ be in the domain of $\mathcal{M}|_{\varphi}$ such that $sR^{\varphi}_{t}t$. We prove $\mathcal{M}|_{\varphi}, t \models \psi$ by induction over the path length from $s$ to $t$. [...]

Announcements and Common Knowledge
Announcements and Common Knowledge

**Proof (ctd.)**

Base case: If the path length is 0, then $s = t$ and $M|s, s \models \psi$, which follows from $M, s \models \chi$, $M, s \models \varphi$, and the validity of $\chi \rightarrow [\varphi]\psi$.

Inductive case: Assume that the path length is $n + 1$ for some $n \in \mathbb{N}$, with $sRaRaBt$ for $a \in B$ and $u \in M|\varphi$. From $M, s \models \chi$, $M, s \models \varphi$, from the validity of $(\chi \land \varphi) \rightarrow E_B \chi$, and $sRaBu$, it follows that $M, u \models \chi$. Because $u$ is in the domain of $M|\varphi$, we know that $M, u \models \varphi$. Now, we can apply the induction hypothesis to the length-$n$ path from $u$ to $t$, which gives us $M|\varphi, t \models \psi$.

**Corollary**

$[\varphi]\psi$ is valid iff $[\varphi]C_B \psi$ is valid.

**Proof.**

$(\leq)$ trivial

$(\geq)$ previous proposition with $\chi = \top$
Unsuccessful Updates

**Question:** Can we characterize successful formulas syntactically?

**Answer:** Not trivially, since it is possible that \( \varphi \) and \( \psi \) are successful, but their conjunction or disjunction are not. (Homework: find such formulas and discuss!)

**Idea for an easy result:** Announcing something that is already public knowledge should not affect existing knowledge. Formally: if we restrict the model in such a way that only "irrelevant" worlds are lost, public knowledge remains public knowledge.

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Unsuccessful Updates

**Definition (Submodel)**

We call a model \( \mathcal{M}' \) a submodel of \( \mathcal{M} \) if \( D(\mathcal{M}') \subseteq D(\mathcal{M}) \) and \( R \) and \( V \) are restricted accordingly.

**Proposition (Public knowledge updates are successful)**

Let \( \varphi \in L_{KC}^{\square} \). Then \([C_A \varphi]C_A \varphi\) is valid.

**Proof sketch.**

Let \((\mathcal{M}, s)\) be arbitrary and assume that \( \mathcal{M}, s \models C_A \varphi \). Then \( \mathcal{M}, t \models \varphi \) and even \( \mathcal{M}, t \models C_A \varphi \) for all \( t \) with \( sR^* A t \). The \( R^* A \)-reachable submodels of \( \mathcal{M}|_{C_A \varphi} = \mathcal{M}|_{\varphi} \) are identical. Hence \( \mathcal{M}|_{C_A \varphi}, s \models C_A \varphi \), i.e., \( \mathcal{M}, s \models [C_A \varphi]C_A \varphi \).

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Unsuccessful Updates

**Question:** What if \( B \subseteq A \)? Is \([C_B \varphi]C_B \varphi\) still valid?

**Answer:** It is not!

**Counterexample:** Recall the example from earlier that showed that \( [p \wedge \neg K_B p]([p \wedge \neg K_B p]) \) is not valid. Let \( B = \{ a \} \). Now consider the update formula \([C_B (p \wedge \neg K_B p)]C_B (p \wedge \neg K_B p)\). This is not valid, obviously.

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Unsuccessful Updates

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Unsuccessful Updates

**Back to the previous positive result (public knowledge updates are successful):** Let us try to generalize the idea of preservation of truth under submodels.

**Definition**

The language \( L_{KC}^0 \) is the following fragment of \( L_{KC} \):

\[
\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid K_A \varphi \mid C_B \varphi \mid [\neg \varphi] \varphi.
\]

**Definition**

A formula \( \varphi \) is preserved under submodels iff, for all \( (\mathcal{M}, s) \) and all submodels \( \mathcal{M}' \) of \( \mathcal{M} \) with \( s \in D(\mathcal{M}') \), if \( \mathcal{M}, s \models \varphi \), then also \( \mathcal{M}', s \models \varphi \).
Proposition (Preservation)

Fragment $L_{K^C}^0$ is preserved under submodels.

Proof.

By structural induction.

- **Base cases:** $p$ and $\neg p$ are trivial: assume that $M'$ is a submodel of $M$ with $s \in D(M')$. Then $M', s \models p$ iff $M, s \models p$.

- **Inductive case $\varphi \land \psi$:**
  
  $M, s \models \varphi \land \psi$ iff
  $M, s \models \varphi$ and $M, s \models \psi$ (2 $\times$ I.H.)

- **Inductive case $\varphi \lor \psi$:**
  
  $M', s \models \varphi$ and $M', s \models \psi$ iff
  $M', s \models \varphi \lor \psi$

- **Inductive case $\neg \varphi$:**

  Suppose for contradiction that $M', s \not\models \neg \varphi$. Then $M', s \models \varphi$. Using the contrapositive of the induction hypothesis, we arrive at $M, s \models \neg \varphi$. Moreover, $M'|\neg \varphi$ is a submodel of $M|\neg \varphi$, because $t \in S'$ only survives if $M', t \models \neg \varphi$. Again by induction hypothesis, $M, t \models \neg \varphi$, so $[\neg \varphi], M' \subseteq [\neg \varphi], M$. But from $M, s \models [\neg \varphi] \land M, s \models \neg \varphi$ it follows that $M|\neg \varphi, s \models \psi$, therefore, by induction hypothesis, $M'|\neg \varphi, s \models \psi$, which is a contradiction.

**Homework:** What about formulas of the form $\tilde{K}_a \varphi$, or $[\varphi] \psi$? Are they also preserved under submodels? If not, why not? Counterexamples?
Unsuccessful Updates

Corollary
Let $\varphi \in \mathcal{L}_{KC}^0$. Then $\varphi \rightarrow [\varphi] \varphi$ is valid.

Proof.
Previous proposition with $\psi = \varphi$.

Corollary ($\mathcal{L}_{KC}^0$ formulas are successful)
Let $\varphi \in \mathcal{L}_{KC}^0$. Then $[\varphi] \varphi$ is valid.

Proof.
Previous corollary using equivalence of $\varphi \rightarrow [\varphi] \varphi$ and $[\varphi] \varphi$.

Remark: The converse does not hold, i.e., there are also formulas not in $\mathcal{L}_{KC}^0$ that are successful. Example: $\neg K a p$.

Or:

Proposition
Inconsistent formulas are successful.

Example
$[p \land \neg p] (p \land \neg p)$

Axiomatisation

Notation:
- $PA$: The set of all valid $\varphi \in \mathcal{L}_{[]}$.  
- $PAC$: The set of all valid $\varphi \in \mathcal{L}_{KC[]}$.  
- $PA$: Axiomatization of $\mathcal{L}_{[]}$ validities (to be defined below)  
- $PAC$: Axiomatization of $\mathcal{L}_{KC[]}$. validities (to be defined below)
Axiomatisation

**PA**

Axioms and inference rules for logic $L_{K[a]}$ with $a \in A$ and $p \in P$:

- all instantiations of propositional tautologies (Taut.)
- $K_a(\varphi \rightarrow \psi) \rightarrow (K_a \varphi \rightarrow K_a \psi)$ (Distribution of $K_a$ over $\rightarrow$)
- $K_a \varphi \rightarrow \varphi$ (Truth)
- $K_a \varphi \rightarrow K_a K_a \varphi$ (Positive introspection)
- $\neg K_a \varphi \rightarrow K_a \neg K_a \varphi$ (Negative introspection)
- $[\varphi][p] \leftrightarrow (\varphi \rightarrow [p \varphi])$ (Atomic permanence)
- $[\varphi](\psi \land \chi) \leftrightarrow ([\varphi] \psi \land [\varphi] \chi)$ (Announcement + conj.)
- $[\varphi]K_a \psi \leftrightarrow ([\varphi] \rightarrow K_a [\varphi] \psi)$ (Announcement + knowledge)
- $[\varphi][\psi] \chi \leftrightarrow [\varphi]([\varphi] \psi) \chi$ (Composition of announcements)
- From $\varphi$ and $\varphi \rightarrow \psi$, infer $\psi$. (Modus ponens)
- From $\varphi$, infer $K_a \varphi$. (Necessitation)

**Example**

We want to show that $\vdash [p]K_a \varphi$:

1. $p \rightarrow p$ (prop. taut.)
2. $[p]p \leftrightarrow (p \rightarrow p)$ (atomic permanence)
3. $[p]p$ (1, 2, another prop. tautology, MP)
4. $K_a[p]p$ (3, necessitation)
5. $p \rightarrow K_a[p]p$ (4, prop. taut.)
6. $[p]K_a[p]p \leftrightarrow (p \rightarrow K_a[p]p)$ (announcements + knowledge)
7. $[p]K_a[p]p$ (5, 6, prop. taut.)

**Theorem**

The axiomatisation $PA$ of $PA$ is sound and complete.

**Note:** We already showed that the axioms involving announcements are sound.
Axiomatisation

PAC

Axioms and inference rules for logic $\mathcal{L}_{KC}] (B \subseteq A)$:
- all axioms and inference rules of $\mathcal{L}_K$
- $C_B(\varphi \rightarrow \psi) \rightarrow (C_B \varphi \rightarrow C_B \psi)$ (Distribution of $C_B$ over $\rightarrow$)
- $C_B \varphi \rightarrow (\varphi \wedge E_B C_B \varphi)$ (Mix)
- $C_B(\varphi \rightarrow E_B \varphi) \rightarrow (\varphi \rightarrow C_B \varphi)$
  (Induction of common knowledge)
- From $\varphi$, infer $C_B \varphi$.
  (Necessitation of common knowledge)
- From $\varphi$, infer $[\psi] \varphi$.
  (Necessitation of announcements)
- From $\chi \rightarrow [\varphi] \psi$ and $\chi \wedge \varphi \rightarrow E_B \chi$, infer $\chi \rightarrow [\varphi] C_B \psi$.
  (Mix of announcements and common knowledge)

Note:
- We already showed soundness for (most of) the additional rules and axioms.

Example: Muddy Children
**Example (Muddy children)**

- There are *n* children. Some of them have a muddy forehead.
- They only see whether the other children are muddy, not themselves.
- They are perfect reasoners/logicians.
- Their father says (repeatedly): “At least one of you is muddy. Those of you who know whether they are muddy please raise your hand.”

**Announcements:** Raising hands or not.

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We look at example with three children (a, b, and c), where a and b are muddy, while c not, i.e., \( m_a \land m_b \land \neg m_c \).

**Some abbreviations:**

- \( \text{muddy} = m_a \lor m_b \lor m_c \).
- \( \text{knowmuddy} = (K_a m_a \lor K_a \neg m_a) \lor (K_b m_b \lor K_b \neg m_b) \lor (K_c m_c \lor K_c \neg m_c) \).
- \( \text{abknowmuddy} = (K_a m_a \lor K_a \neg m_a) \land (K_b m_b \lor K_b \neg m_b) \).

---

**Model Cube**

\( \text{Model Cube}^{\prime \prime \prime} = \text{Cube}^{\prime \prime} | \text{abknowmuddy} \) (after a and b raise their hands):

- Cube\( \prime \prime \), 110 \models \text{knowmuddy}
- Cube\( \prime \prime \), 110 \models C_{abc}(m_a \land m_b \land \neg m_c)

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**Summary**
Public announcements change knowledge state.

Semantics: via submodels

Without common knowledge: $L_{K||}$ can be reduced to $L_K$.

With common knowledge: not.

Announcements can be successful or unsuccessful.
Preserved formulas are successful.

Sound and complete axiomatizations exist.