Principles of Knowledge Representation and Reasoning
Belief Revision

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Introduction
Belief change

A dual approach to nonmonotonic reasoning is belief change.

We start with some belief state $K$. When new information arrives, we change the belief state in order to accommodate the new information.

In the general case, the changed belief state may not be a superset of the original belief state.

Contrary to nonmonotonic reasoning, here we deal with temporal nonmonotonicity, i.e., the nonmonotonic evolution of a knowledge base or belief state over time.
Two scenarios

- We have a theory about the world, and the new information is meant to **correct** our theory …

  - Belief revision: change your belief state minimally in order to accommodate the new information
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  \Rightarrow \textbf{Belief revision: change your belief state minimally in order to accommodate the new information}
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- We have a correct theory about the current state of the world, and the new information is meant to record a **change** in the world ...

  \[
  \Rightarrow \textbf{Belief update: incorporate the change by assuming that the world has changed minimally}
  \]
Update and revision are different

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- For **belief update** this is not necessarily the case.
  - Assume we know that the **door is open or the window is open**.
  - Assume we learn that the world has changed and the **door is now closed**.
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- In case of belief revision, we would like to add the new information monotonically to our old beliefs.
- For belief update this is not necessarily the case.
  - Assume we know that the door is open or the window is open.
  - Assume we learn that the world has changed and the door is now closed.
- In this case, we do not want to add this information monotonically to our theory, since we would be forced to conclude that the window is open.
Belief Revision
Belief revision

- How to react to new information? $K$ is the knowledge base, $A$ some new information

$K \cup A \rightarrow$ inconsistency
Belief revision

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- Union → inconsistency
- Accept loss of beliefs
Belief revision

- How to react to new information? $K$ is the knowledge base, $A$ some new information

- Union $\rightarrow$ inconsistency
- Accept loss of beliefs
- $A$ has priority over $K$
- Saving the most from $K$
Belief change operations

General assumption:

- A belief state is modeled by a deductively closed theory, i.e., $K = Cn(K)$ with $Cn$ the consequence operator.
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- \( \mathcal{L} \): logical language (propositional logic)
Belief change operations

General assumption:
- A belief state is modeled by a deductively closed theory, i.e., $K = \text{Cn}(K)$ with Cn the consequence operator
- $\mathcal{L}$: logical language (propositional logic)
- $\text{Th}_{\mathcal{L}}$: the set of all deductively closed theories (called belief sets) over $\mathcal{L}$

Belief change operations

Most belief change operations have the form:

$$\text{op}: \text{Th}_{\mathcal{L}} \times \mathcal{L} \to \text{Th}_{\mathcal{L}}$$

- Expansion: $K + \psi := \text{Cn}(K \cup \{\psi\})$
- Revision: $K + \varphi$
- Contraction: $K - \varphi$ (removal of some belief)
Revision vs Contraction

How are revision and contraction related to each other?

- **Levi identity**
  \[ K + \phi \equiv (K - \neg \phi) + \phi \]

- **Harper identity**
  \[ K - \phi \equiv K \cap (K + \neg \phi) \]
Revision vs Contraction

How are revision and contraction related to each other?

Given a contraction operator, one can define a revision operator:

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**Harper identity**

\[ K \div \varphi \equiv K \cap (K + \neg \varphi) \]
What is a good revision operator?

Rationale of revision operator:

- **Consistency**: a revision has to produce a consistent set of beliefs
- **Minimality** of change: a revision has to change as few beliefs as possible
- **Priority** to the new information: the ’new’ information is considered more important than the ’old’ one
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To characterize rational revision operators, Alchourron, Gärdenfors, and Makinson identified conditions that should be satisfied by such an operator.
AGM Postulates: Constraining the space of revision operations

AGM postulates:

(+1) \( K + \varphi \in \text{Th}_L \);
(+2) \( \varphi \in K + \varphi \);
(+3) \( K + \varphi \subseteq K + \varphi \);
(+4) If \( \neg \varphi \notin K \), then \( K + \varphi \subseteq K + \varphi \);
(+5) \( K + \varphi = \text{Cn}(\bot) \) only if \( \vdash \neg \varphi \);
(+6) If \( \vdash \varphi \leftrightarrow \psi \) then \( K + \varphi = K + \psi \);
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(+5) $K + \varphi = \text{Cn}(\bot)$ only if $\vdash \neg \varphi$;
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Supplementary postulates:

(+7) $K + (\varphi \land \psi) \subseteq (K + \varphi) + \psi$;
(+8) If $\neg \psi \notin K + \varphi$, then $(K + \varphi) + \psi \subseteq K + (\varphi \land \psi)$.
Canonical revision operations?

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- The postulates constrain the space to fully rational revision operations, but do not pick a single one.

- Revision operations are closed under intersection, so should we choose the minimum?
Remainder set

Given a belief set $K$ and some new information $\varphi$, we are specifically interested in the maximal subtheories consistent with $\varphi$:

**Definition**

Let $A \cup \{\varphi\}$ be a set of formulae. The $\varphi$-remainder set of $A$, denoted by $A \perp \varphi$, is the set of all (inclusion-) maximal subsets $B$ of $A$ that do not entail $\varphi$, i.e.:

1. $\varphi \notin Cn(B)$
2. There is no set $B'$ such that $B \subsetneq B' \subseteq A$ with $\varphi \notin Cn(B')$
Canonical revision operations: Full-meet revision

### Full-meet contraction/revision

- **Full-meet contraction:** \( K \div \phi = \bigcap (K \perp \phi) \) (if \( K \perp \phi \neq \emptyset \); \( = K \), else)
- **Full-meet revision:** \( K + \phi = (K \div \neg \phi) + \phi \).

- Is full-meet contraction reasonable?
Canonical revision operations: Full-meet revision

Full-meet contraction/revision

Full-meet contraction: $K \div \varphi = \bigcap (K \perp \varphi)$ (if $K \perp \varphi \neq \emptyset$; $= K$, else)

Full-meet revision: $K + \varphi = (K \div \neg \varphi) + \varphi$.

- Is full-meet contraction reasonable?
- Easy to show: all AGM postulates are satisfied.
- But: it is far too cautious.
  - Given $\varphi$ is inconsistent with $K$, we get: $K + \varphi = \text{Cn}(\varphi)$
  - More reasonable: define contraction by only considering some of the remainders: $\rightsquigarrow$ partial meet contraction
- Are there other revision schemes?
Belief revision schemes

- Preference information (what to keep and what to give up)
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- In general, a belief revision scheme (BRS) is a “recipe” for deriving a revision operation – restricted to a particular set $K$ – from
  - the belief set and
Belief revision schemes

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- In general, a belief revision scheme (BRS) is a “recipe” for deriving a revision operation – restricted to a particular set $K$ – from
  - the belief set and
  - preference information over this belief set
Examples

Partial meet revision (AGM): Preference information is given by a selection function $\gamma$ over the set of maximal subtheories consistent with the new information:

$$K + \varphi \overset{\text{def}}{=} \left( \bigcap \gamma(K \perp \neg \varphi) \right) + \varphi.$$
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Cut revision (GM): Preference information is given by a complete preorder $\preceq$ over all $\psi \in K$:

$$K + \varphi \overset{\text{def}}{=} \{ \psi \in K \mid \neg \varphi \prec \psi \} + \varphi.$$

Provided $\preceq$ satisfies a number of axioms (epistemic entrenchment), cut revisions correspond to fully rational revision operations.
Revision – Viewed computationally

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- Consider belief bases (finite sets of propositions) to represent belief sets.
Revision – Viewed computationally

- We don’t want to deal with deductively closed theories . . .

- Consider **belief bases** (finite sets of propositions) to **represent** belief sets.

- We don’t want to specify an arbitrary amount of preference information . . .

- A theory $K$ over the propositional logic $\mathcal{L}$ with $n$ propositional atoms can have as much as
  - $2^{2^n}$ different propositions,
  - $2^n$ different models.

- Consider ways of specifying preference information in a **concise** way, i.e., polynomial in the size of the belief base.
Base revision schemes

- Start with a finite belief base \( A \) and preference information over the elements of \( A \) …
- We want to generate a revision operation (restricted to \( C_n(A) \))
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We want to generate a revision operation (restricted to $C_n(A)$)

Assume a partitioning of $A$ into $n$ priority classes $A_1, \ldots, A_n$ such that the elements of $A_i$ are more important or relevant than those of $A_j$ for $j < i$
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Base revision schemes generate revision operations in the same way as ordinary schemes do.
Example: Prioritized Meet-Base Revision

Let \((A \downarrow \varphi)\) be the maximal subsets of \(A\) that are consistent with \(\neg \varphi\) and maximize relevant propositions.
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**Definition**

Let \(A \cup \{\varphi\}\) be a set of formulae. The **prioritized base-removal** \(A \downarrow \varphi\) is the set of all subsets \(B\) of \(A\) such that:

1. \(\varphi \notin Cn(B)\)
2. For each \(C \subseteq A\) and \(1 \leq j \leq n\), if \(B \cap \bigcup_{i \geq j} A_i \varsubsetneq C \cap \bigcup_{i \geq j} A_i\), then \(\varphi \in Cn(C \cap \bigcup_{i \geq j} A_i)\).

Note that the 2nd condition is equivalent to:
For each \(1 \leq j \leq n\) and each \(C \subseteq \bigcup_{i \geq j} A_i\), if \(B \cap \bigcup_{i \geq j} A_i \varsubsetneq C\), then \(\varphi \in Cn(C)\).
Prioritized Meet-Base Revision (PMBR):

\[ A \oplus \varphi \overset{\text{def}}{=} \left( \bigcap_{B \in (A \downarrow \neg \varphi)} \text{Cn}(B) \right) + \varphi. \]
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Prioritized Meet-Base Revision (PMBR):

\[ A \oplus \varphi \triangleq \left( \bigcap_{B \in (A \downarrow \neg \varphi)} \text{Cn}(B) \right) + \varphi. \]

Define a revision operation \( \dagger \) on \( \text{Cn}(A) \) (that depends on \( A \) and the priority information) by

\[ \text{Cn}(A) \dagger \varphi \triangleq A \oplus \varphi. \]
Properties of PMBRs

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- A revised base can become **exponentially large**:
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  A = \{p_1, \ldots, p_m, q_1, \ldots, q_m\}, \quad \varphi = \bigwedge_{i=1}^{m} (p_i \leftrightarrow \neg q_i)
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  \((A \Downarrow \varphi)\) has size exponential in \(|A|\).
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\((A \downarrow \varphi)\) has size exponential in \(|A|\).
- **Worse**, in some cases there exists no concise representation of the revised base (provided the polynomial hierarchy does not collapse [Cadoli et al 94]).
Revision vs. Nonmonotonic Reasoning

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- Given $K$ and a revision operation $+$, a nonmonotonic consequence relation can be defined as follows: $\varphi \vdash \psi$ iff $\psi \in K + \varphi$.

In this case,

- the rationality postulates correspond to principles of NMR (such as cautious monotonicity, etc.).
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- in the case of prerequisite-free, normal defaults $D$, the cautious conclusions from $(W, D)$ are simply $D \oplus W$ with one priority level;
- a similar relationship holds between Brewka’s level default theories and PMBRs.
NMR Principles and Rationality Postulates

\[ +2 \quad \varphi \in K + \varphi; \]

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(+6) If \( \vdash \varphi \leftrightarrow \psi \) then \( K + \varphi = K + \psi; \)

- **Left Logical Equivalence**
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- Left Logical Equivalence

(+8) If \( \neg \psi \notin K + \varphi, \)
then \( (K + \varphi) + \psi \subseteq K + (\varphi \land \psi); \)

- Rational Monotonicity
Conclusions from the Correspondence

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Conclusions from the Correspondence

- NMR can be thought of as the other side of the same coin.
- NMR (at least for default logic) is as hard as belief revision.
- Representing the conclusions from a propositional default theory using classical propositional logic cannot be done in polynomial space, provided the polynomial hierarchy does not collapse.
- In other words, nonmonotonic logics can be thought of representing (some) information in a denser way than classical logic, and with that come higher computational costs.
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AGM Postulates
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Priorities
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Conclusions
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Outlook & Summary

While NMR and Belief Revision seem to be the two sides of the same coin, there are notable **pragmatic differences**:

- Belief revision seems to require that we can easily represent the changed belief base, while for NMR it makes sense to use **dense representations**.
- A similar argument could be made for the **computational complexity**.
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- NMR and Belief Revision can be thought of as **qualitative ways** of dealing with uncertainty in a purely logical setting.

- There exists a strong **correspondence** between NMR and Belief Revision.

- Both are computationally expensive and representational problematic.

- There are cases, though, that are tractable and practical.
Literature
**Literature I**

- **D. Makinson.**
  How to give it up: A survey of some formal aspects of theory change. 
  
  Very good introduction to the topic.

- **B. Nebel.**
  Belief revision and default reasoning: Syntax-based approaches. 

- **C. E. Alchourrón, P. Gärdenfors, and D. Makinson.**
  On the logic of theory change: Partial meet contraction and revision functions. 
  
  Introduces the so-called AGM approaches: Characterizing belief revision operations by postulates.

- **P. Gärdenfors.**
  *Knowledge in Flux—Modeling the Dynamics of Epistemic States.* 
B. Nebel.
How hard is it to revise a belief base?

P. Gärdenfors.
Belief revision and nonmonotonic logic: Two sides of the same coin?
ECAI-90, 768-773.

H. Rott.
Change, choice and inference: A study of belief revision and nonmonotonic reasoning,

P. Peppasa, Mary-Anne Williams, Samir Chopra, and Norman Foo
Relevance in belief revision