Principles of Knowledge Representation and Reasoning

Answer Set Programming

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Introduction

ASP: Background

- Answer set semantics: a formalization of negation as failure in logic programming (Prolog)
- Several formal semantics: well-founded semantics, perfect-model semantics, inflationary semantics, ...
- Can be viewed as a simpler variant of default logic

ASP: Negation as failure

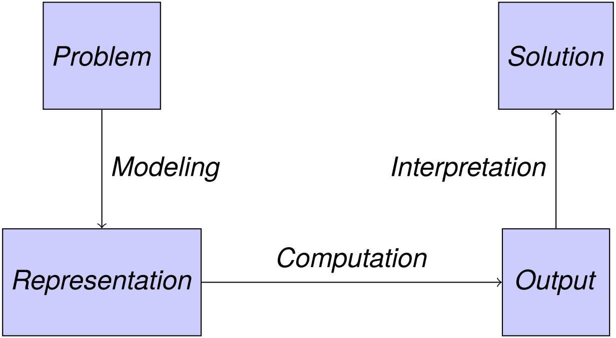
- Another interpretation for negation: $\text{not}\,x \equiv$ "It cannot be shown that x is true"
- For example, you are innocent until proven guilty

Example

$$\mathit{innocent} \leftarrow \text{not}\,\mathit{guilty}.$$

ASP: Declarative problem solving

- What is the problem? instead of: How to solve the problem?
- Outsourcing the computation part to an external solver



Answer Sets

Normal logic programs I

Let $\mathcal{A}$ be a set of first-order atoms.

Rules:

$$a \leftarrow b_1, \ldots, b_m, \text{not}\, c_1, \ldots, \text{not}\, c_k$$

where $\{a, b_1, \ldots, b_m, c_1, \ldots, c_k\} \subseteq \mathcal{A}$

- Meaning similar to default logic:
  If
  1. **we have derived** $b_1, \ldots, b_m$ **and**
  2. **cannot derive any of** $c_1, \ldots, c_k$**,**

  then derive a**.**
- Rules without right-hand side (facts): $a \leftarrow$
- Rules without left-hand side (constraints):

$$\leftarrow b_1, \ldots, b_m, \text{not}\, c_1, \ldots, \text{not}\, c_k$$

Normal logic programs II

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where $\{a, b_1, \ldots, b_m, c_1, \ldots, c_k\} \subseteq \mathcal{A}$

- a is called the head of the rule, denoted by $\text{head}(r)$.
- The literals $b_1, \ldots, b_m$ form the positive body of r, denoted by $\text{body}^+(r)$.
- The literals $\text{not}\, c_1, \ldots, \text{not}\, c_k$ form the negative body of r, denoted by $\text{body}^-(r)$.
- The body of r is the union of positive and negative body: $\text{body}(r) = \text{body}^+(r) \cup \text{body}^-(r)$.

Normal logic programs: Example

Example

$$
\begin{aligned}
\mathit{bird}(X) &\leftarrow \mathit{eagle}(X)\\
\mathit{bird}(X) &\leftarrow \mathit{penguin}(X)\\
\mathit{fly}(X) &\leftarrow \mathit{bird}(X), \mathrm{not}\, \mathit{nonfly}(X)\\
\mathit{nonfly}(X) &\leftarrow \mathit{penguin}(X)\\
\mathit{eagle}(\mathit{eddy}) &\leftarrow\\
\mathit{penguin}(\mathit{tweety}) &\leftarrow
\end{aligned}
$$

Herbrand base and grounded rules

Let P be a normal logic program, i.e., a finite set of rules as described above.

- The Herbrand universe (symb. U_P) of P is the set of ground terms constructed from the function symbols and constants in P.
- The Herbrand base of P (symb. B_P) is the set of ground atoms constructed from predicate symbols and ground terms from the Herbrand universe.
- From now on, a program will refer to the set of its grounded rules.
- The set of atoms in P is denoted by $atoms(P)$.

Herbrand base and grounded rules

Example

$$
\begin{aligned}
\mathit{bird}(\mathit{eddy}) &\leftarrow \mathit{eagle}(\mathit{eddy})\\
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Satisfaction relation:

- $X \models a$ if $a \in X$.
- $X \models r$ if $\{b_1, \dots, b_m\} \not\subseteq X$ or $\{a, c_1, \dots, c_n\} \cap X \neq \emptyset$, where $r = a \leftarrow b_1, \dots, b_m, \text{not}\, c_1, \dots, \text{not}\, c_k$.
- $X \models P$ if $X \models r$ for each $r \in P$.

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Idea

Idea: “models” as interpretations that are satisfying, stable, and supported.

Positive (*not-free*) logic programs

Definition (Answer set)

Let P be a logic program without **not**, $X \subseteq \text{atoms}(P)$.
X is the (unique) answer set of P if it is the least fixpoint of the operator:

$$\Gamma_P(X) = \{a \colon \exists r = a \leftarrow b_1, \ldots, b_m \in P \text{ with } \{b_1, \ldots, b_m\} \subseteq X\}.$$

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$$P = \left\{ \begin{array}{lll} a \leftarrow b, & d \leftarrow f, & b \leftarrow\ , \\ d \leftarrow b, & c \leftarrow d, & e \leftarrow f \end{array} \right\}$$

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Gelfond-Lifschitz reduct

Definition (Reduct)

The reduct of a program P with respect to a set of atoms $X \subseteq \text{atoms}(P)$ is defined as:

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$X \subseteq \text{atoms}(P)$ is an answer set of P if X is an answer set of P^X.

Answer sets: Examples

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Proposition

Each answer set of a normal logic program P is a minimal model of P, i.e., it satisfies all rules in P and there is no proper subset of P satisfying all rules in P.

Notice: The converse is not true: not each minimal model is an answer set.

Some properties II

Proposition

Let F be a set of (non-constraint) rules and G be a set of constraints. A set of atoms X is an answer set of $F \cup G$ iff it is an answer set of F that satisfies G.

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Proof.

$F \subseteq F \cup G$ implies $F^X \subseteq (F \cup G)^X$ and hence $\mathrm{lfp}_\Gamma(F^X) \subseteq \mathrm{lfp}_\Gamma((F \cup G)^X))$.

$\Rightarrow$: Assume X is an answer set of $F \cup G$, hence $X = \mathrm{lfp}_\Gamma((F \cup G)^X)$ and $X \models G$. Since G contains constraints only, it follows that each $a \in X$ is the head of some rule in F. Hence, $X \subseteq \mathrm{lfp}_\Gamma(F^X)$, and thus X is an answer set of F that satisfies G.

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$\Leftarrow$: Similar.

□

Complexity: Existence of answer sets is NP-complete

1. Membership in NP: Guess $X \subseteq \text{atoms}(P)$ (**nondet. polytime**), compute P^X, compute its closure, compare to X (**everything det. polytime**).

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2. NP-hardness: Reduction from 3SAT: an answer set exists iff the following clauses are satisfiable:

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for every propositional variable p occurring in the clauses, and

$$\leftarrow \mathrm{not}\, l_1', \mathrm{not}\, l_2', \mathrm{not}\, l_3'$$

for every clause $l_1 \vee l_2 \vee l_3$, where $l_i' = p$ if $l_i = p$ and $l_i' = \hat{p}$ if $l_i = \neg p$.

Difference to Propositional Logic

- The **ancestor** relation is the transitive closure of the **parent** relation.
- Transitive closure cannot be (concisely) represented in propositional/predicate logic.

  $$par(X,Y) \rightarrow anc(X,Y)$$
  $$par(X,Z) \land anc(Z,Y) \rightarrow anc(X,Y)$$

  The above formulae only guarantee that **anc** is a superset of the transitive closure of **par**.
- For transitive closure one needs the minimality condition in some form: nonmonotonic logics, fixpoint logics, ...

Stratification

The reason for multiple answer sets is the fact that *a* may depend on *b* and simultaneously *b* may depend on *a*.
The lack of this kind of circular dependencies makes reasoning easier.

Definition

A logic program P is stratified if P can be partitioned to $P = P_1 \cup \dots \cup P_n$ so that for all $i \in \{1, \dots, n\}$ and $(a \leftarrow b_1, \dots, b_m, \mathrm{not}\, c_1, \dots, \mathrm{not}\, c_k) \in P_i$,

1. there is no $\mathrm{not}\, a$ in P_i and
2. there are no occurrences of a anywhere in $P_1 \cup \dots \cup P_{i-1}$.

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Theorem

A stratified program P has exactly one answer set. The unique answer set can be computed in polynomial time.

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Example

Our earlier examples with more than one or no answer sets:

$$P_3 = \{p \leftarrow \text{not}\, p\}$$
$$P_4 = \{p \leftarrow \text{not}\, q, \quad q \leftarrow \text{not}\, p\}$$

AnsProlog and ASP Tools

Programs for Reasoning with Answer Sets

- smodels (Niemelä & Simons), dlv (Eiter et al.), clasp (Schaub et al.), ...
- Schematic input:

```
p(X) :- not q(X).      anc(X,Y) :- par(X,Y).
q(X) :- not p(X).      anc(X,Y) :- par(X,Z), anc(Z,Y).
r(a).                  par(a,b). par(a,c). par(b,d).
r(b).                  female(a).
r(c).                  male(X) :- not(female(X)).
                       forefather(X,Y) :-
                                  anc(X,Y), male(X).
```

AnsProlog

- Propositions are any combination of lowercase letters.
- Variables are any combination of letters starting with an uppercase letter.
- Write ":-" instead of $\leftarrow$.
- Integers can be used and so can ne arithmetic operations $(+, -, *, /, \%)$.
- Negation as failure is denoted by not.
- Strong negation is denoted by $-$.
- `#const n = ...` statements can be used to define constants.
- The `#hide`/`#show` statements can be used to influence which iterals are shown in the solution.

AnsProlog: Choice functions

- The literal `{b1; ... ; bm}`
  is true iff any subset of the set `{b1,...,bm}` is true.

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```
{ a(1); a(2); a(3) }.
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Generate all interpretations over the atoms $a(1), a(2), a(3)$:

```
{ a(1); a(2); a(3) }.
```

With strong negation:

```
-a(X) :- not a(X), X=1..3.
{ a(1..3) }.
```

AnsProlog: Choice with cardinality

- The literal `l {b1; ...; bm} u`
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Generate all interpretations over the atoms $a(1), a(2), a(3), b(1), b(2)$ that contain exactly 2 true atoms:

```
2 { a(1..3); b(1..2) } 2.
```

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Generate all interpretations over the atoms $a(1), a(2), a(3), b(1), b(2)$, $b(3)$ that do not contain exactly 2 or more true atoms for the same predicate:

```
{ a(1..3); b(1..3) }.
:- 2 { a(1..3) } 3.
:- 2 { b(1..3) } 3.
```

AnsProlog: Domains of variables

- The domain of a variable must be known in order to avoid “unsafe”-error while the program is grounded.
- The domain can be set literal-wise, rule-wise, or program wise.
- For limiting the scope within a literal use the syntax:
  `a(X) : dom(X)` or `a(X) : X=1..3`

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Example

```
num(0..10).
even(2*X) :- num(X), 2*X <=10.
1 { a(X) : even(X) } 1.

#show a/1.
```

Example: Graph coloring

Example

```
#const n = 2.
c(1..n).
1 {color(X,I) : c(I)} 1 :- v(X).
:- color(X,I), color(Y,I), e(X,Y), c(I).

% Instance
v(1..4).
e(1,2).
e(1,3).
e(2,4).
e(3,4).
% e(2,3).

#show color/2.
```

Generate and test

ASP programs are often organized in a “generate-and-test” style: first describe candidate solutions, then rule out possible solutions by stating constraints.

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Example

```
% n-Queens encoding %
#const n = 4.

% Generate possible positions %
1 { q(I,1..n) } 1 :- I = 1..n.

% Rule out attacking positions %
:- q(I1,J), q(I2,J), I1 != I2.
:- q(I,J), q(I+D,J+D), D = 1..n.
:- q(I,J), q(I+D,J-D), D = 1..n.
```

Generate and test: Further example

Problem: In a graph find cliques of size $\geq n$

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Example

```
#const n = 3.

edge(X,Y) :- edge(Y,X).
n {clique(X) : node(X)}.
:- clique(X), clique(Y), node(X), node(Y), X!=Y, not edge(X,Y).

% Instance %
node(1..5).
edge(1,2;4).
edge(2,3;4).
edge(3,4).
edge(4,2;5).

#show clique/1.
```

AnsProlog: Miscellaneous

The language is even bigger than that! It includes

- Disjunction in the head
- Other operators: `#sum`,`#min`,`#max`,`#even`,`#odd`,`#avg`, ...
- Multi-criteria optimizations
- Heuristic optimizations
- ...

(More on that in the exercises!)

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