Principles of
Knowledge Representation and Reasoning
Answer Set Programming

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1 Introduction
Answer set semantics: a formalization of negation as failure in logic programming (Prolog)

Several formal semantics: well-founded semantics, perfect-model semantics, inflationary semantics, ...

Can be viewed as a simpler variant of default logic
ASP: Negation as failure

- Another interpretation for negation: \( \text{not} \, x \equiv \text{"It cannot be shown that } x \text{ is true"} \)
- For example, you are innocent until proven guilty

Example

\[ \text{innocent} \leftarrow \text{not guilty}. \]
ASP: Declarative problem solving

- What is the problem? instead of: How to solve the problem?
- Outsourcing the computation part to an external solver
2 Answer Sets

- Normal logic programs
- Interpretation and Satisfiability
- Definition
- Formal properties
- Stratification
Normal logic programs I

Let $\mathcal{A}$ be a set of first-order atoms.

**Rules:**

\[
a \leftarrow b_1, \ldots, b_m, \text{not } c_1, \ldots, \text{not } c_k
\]

where $\{a, b_1, \ldots, b_m, c_1, \ldots, c_k\} \subseteq \mathcal{A}$

- Meaning similar to default logic:
  - If
    1. we have derived $b_1, \ldots, b_m$ and
    2. cannot derive any of $c_1, \ldots, c_k$,
  - then derive $a$.
- Rules without right-hand side (facts): $a \leftarrow$
- Rules without left-hand side (constraints):
  \[
  \leftarrow b_1, \ldots, b_m, \text{not } c_1, \ldots, \text{not } c_k
  \]
Normal logic programs II

Let $\mathcal{A}$ be a set of first-order atoms.

Rules:

\[
a \leftarrow b_1, \ldots, b_m, \text{not } c_1, \ldots, \text{not } c_k
\]

where $\{a, b_1, \ldots, b_m, c_1, \ldots, c_k\} \subseteq \mathcal{A}$

- $a$ is called the head of the rule, denoted by $\text{head}(r)$.
- The literals $b_1, \ldots, b_m$ form the positive body of $r$, denoted by $\text{body}^+(r)$.
- The literals not $c_1, \ldots, \text{not } c_k$ form the negative body of $r$, denoted by $\text{body}^-(r)$.
- The body of $r$ is the union of positive and negative body: $\text{body}(r) = \text{body}^+(r) \cup \text{body}^-(r)$. 
Normal logic programs: Example

Example

\[
\begin{align*}
  bird(X) & \leftarrow eagle(X) \\
  bird(X) & \leftarrow penguin(X) \\
  fly(X) & \leftarrow bird(X), \text{not} nonfly(X) \\
  nonfly(X) & \leftarrow penguin(X) \\
  eagle(\text{eddy}) & \leftarrow \\
  penguin(\text{tweety}) & \leftarrow
\end{align*}
\]
Let $P$ be a normal logic program, i.e., a finite set of rules as described above.

- The **Herbrand universe** (symb. $U_P$) of $P$ is the set of ground terms constructed from the function symbols and constants in $P$.

- The **Herbrand base** of $P$ (symb. $B_P$) is the set of ground atoms constructed from predicate symbols and ground terms from the Herbrand universe.

- From now on, a program will refer to the set of its grounded rules.

- The set of atoms in $P$ is denoted by $atoms(P)$.
Herbrand base and grounded rules

Example

\begin{align*}
\text{bird}(\text{eddy}) & \leftarrow \text{eagle}(\text{eddy}) \\
\text{bird}(\text{tweety}) & \leftarrow \text{eagle}(\text{tweety}) \\
\text{bird}(\text{eddy}) & \leftarrow \text{penguin}(\text{eddy}) \\
\text{bird}(\text{tweety}) & \leftarrow \text{penguin}(\text{tweety}) \\
\text{fly}(\text{eddy}) & \leftarrow \text{bird}(\text{eddy}), \neg \text{nonfly}(\text{eddy}) \\
\text{fly}(\text{tweety}) & \leftarrow \text{bird}(\text{tweety}), \neg \text{nonfly}(\text{tweety}) \\
\text{nonfly}(\text{eddy}) & \leftarrow \text{penguin}(\text{eddy}) \\
\text{nonfly}(\text{tweety}) & \leftarrow \text{penguin}(\text{tweety}) \\
\text{eagle}(\text{eddy}) & \leftarrow \\
\text{penguin}(\text{tweety}) & \leftarrow
\end{align*}
Satisfaction

A Herbrand interpretation is a subset $X$ of the Herbrand base.

Satisfaction relation:

- $X \models a$ if $a \in X$.
- $X \models r$ if $\{b_1, \ldots, b_m\} \not\subseteq X$ or $\{a, c_1, \ldots, c_n\} \cap X \neq \emptyset$, where $r = a \leftarrow b_1, \ldots, b_m$, not $c_1, \ldots$, not $c_k$.
- $X \models P$ if $X \models r$ for each $r \in P$.

Idea

Idea: “models” as interpretations that are satisfying, stable, and supported.
Positive *(not-free)* logic programs

**Definition (Answer set)**

Let $P$ be a logic program without *not*, $X \subseteq \text{atoms}(P)$. $X$ is the (unique) answer set of $P$ if it is the least fixpoint of the operator:

$$\Gamma_P(X) = \{a : \exists r = a \leftarrow b_1, \ldots, b_m \in P \text{ with } \{b_1, \ldots, b_m\} \subseteq X\}.$$

**Example**

\begin{align*}
P = \{ & a \leftarrow b, \quad d \leftarrow f, \quad b \leftarrow, \\
& d \leftarrow b, \quad c \leftarrow d, \quad e \leftarrow f \}
\end{align*}

\begin{align*}
\Gamma^0 = \emptyset, \quad \Gamma^1 = \Gamma(\emptyset) = \{b\}, \quad \Gamma^2 = \Gamma(\Gamma^1) = \{b, d, a\}, \quad \Gamma^3 = \\
\Gamma(\Gamma^2) = \{b, d, a, c\}, \quad \Gamma^4 = \Gamma(\Gamma^3) = \{b, d, a, c\} = \Gamma^3
\end{align*}
Gelfond-Lifschitz reduct

Definition (Reduct)

The reduct of a program $P$ with respect to a set of atoms $X \subseteq \text{atoms}(P)$ is defined as:

$$P^X := \{ \text{head}(r) \leftarrow \text{body}^+(r) : r \in P, \text{ c } \not\in X \text{ for each not } c \in \text{body}^-(r) \}$$

That is, given $X$,

- ... delete all rules whose negative part contradicts $X$
- ... remove all negated atoms from the remaining rules

Definition (Answer set)

$X \subseteq \text{atoms}(P)$ is an answer set of $P$ if $X$ is an answer set of $P^X$. 
Answer sets: Examples

Example

\[ a \leftarrow \neg b, \quad b \leftarrow \neg a, \]
\[ c \leftarrow a, \quad d \leftarrow b. \]

Example

\[ a \leftarrow \neg b, \quad b \leftarrow \neg a, \]
\[ b \leftarrow a, \quad c \leftarrow b \]

Example

\[ a \leftarrow b, \quad b \leftarrow a \]
Some properties I

Proposition

*If an atom a belongs to an answer set of a logic program P, then a is the head of one of the rules of P.*

Proposition

*Each answer set of a normal logic program P is a minimal model of P, i.e., it satisfies all rules in P and there is no proper subset of P satisfying all rules in P.*

Notice: The converse is not true: not each minimal model is an answer set.
Some properties II

Proposition

Let $F$ be a set of (non-constraint) rules and $G$ be a set of constraints. A set of atoms $X$ is an answer set of $F \cup G$ iff it is an answer set of $F$ that satisfies $G$.

Proof.

$F \subseteq F \cup G$ implies $F^X \subseteq (F \cup G)^X$ and hence $\text{lfp}_\Gamma(F^X) \subseteq \text{lfp}_\Gamma((F \cup G)^X))$.

$\Rightarrow$: Assume $X$ is an answer set of $F \cup G$, hence $X = \text{lfp}_\Gamma((F \cup G)^X)$ and $X \models G$. Since $G$ contains constraints only, it follows that each $a \in X$ is the head of some rule in $F$. Hence, $X \subseteq \text{lfp}_\Gamma(F^X)$, and thus $X$ is an answer set of $F$ that satisfies $G$.

$\Leftarrow$: Similar.
Complexity: Existence of answer sets is NP-complete

1. **Membership in NP**: Guess \( X \subseteq \text{atoms}(P) \) (\text{nondet. polytime}), compute \( P^X \), compute its closure, compare to \( X \) (everything det. polytime).

2. **NP-hardness**: Reduction from 3SAT: an answer set exists iff the following clauses are satisfiable:

\[
p \leftarrow \text{not} \hat{p}. \quad \hat{p} \leftarrow \text{not} p.
\]

for every propositional variable \( p \) occurring in the clauses, and

\[
\text{not} l_1', \text{not} l_2', \text{not} l_3'
\]

for every clause \( l_1 \lor l_2 \lor l_3 \), where \( l_i' = p \) if \( l_i = p \) and \( l_i' = \hat{p} \) if \( l_i = \text{not} p \).
Difference to Propositional Logic

- The **ancestor** relation is the **transitive closure** of the **parent** relation.
- Transitive closure **cannot be** (concisely) represented in propositional/predicate logic.
  
  \[
  par(X, Y) \rightarrow \text{anc}(X, Y) \\
  par(X, Z) \land \text{anc}(Z, Y) \rightarrow \text{anc}(X, Y)
  \]

  The above formulae only guarantee that **anc** is a **superset** of the transitive closure of **par**.

- For transitive closure one needs the **minimality condition** in some form: nonmonotonic logics, fixpoint logics, ...
The reason for multiple answer sets is the fact that $a$ may depend on $b$ and simultaneously $b$ may depend on $a$. The lack of this kind of circular dependencies makes reasoning easier.

**Definition**

A logic program $P$ is **stratified** if $P$ can be partitioned to $P = P_1 \cup \cdots \cup P_n$ so that for all $i \in \{1, \ldots, n\}$ and $(a \leftarrow b_1, \ldots, b_m, \neg c_1, \ldots, \neg c_k) \in P_i$,

1. there is no $\neg a$ in $P_i$ and
2. there are no occurrences of $a$ anywhere in $P_1 \cup \cdots \cup P_{i-1}$.
### Stratification

#### Theorem

A stratified program $P$ has exactly one answer set. The unique answer set can be computed in polynomial time.

#### Example

Our earlier examples with more than one or no answer sets:

$$P_3 = \{ p \leftarrow \text{not } p \}$$

$$P_4 = \{ p \leftarrow \text{not } q, \quad q \leftarrow \text{not } p \}$$
3 AnsProlog and ASP Tools

- Language and notations
Programs for Reasoning with Answer Sets

- smodels (Niemelä & Simons), dlv (Eiter et al.), clasp (Schaub et al.), ...

- Schematic input:

```prolog
p(X) :- not q(X).
q(X) :- not p(X).
r(a).
r(b).
r(c).
anc(X,Y) :- par(X,Y).
anc(X,Y) :- par(X,Z), anc(Z,Y).
par(a,b). par(a,c). par(b,d).
female(a).
male(X) :- not(female(X)).
forefather(X,Y) :-
    anc(X,Y), male(X).
```
AnsProlog

- Propositions are any combination of lowercase letters.
- Variables are any combination of letters starting with an uppercase letter.
- Write ":-" instead of ←.
- Integers can be used and so can arithmetic operations (+, −, *, /, %).
- **Negation as failure** is denoted by not.
- **Strong negation** is denoted by ¬.
- `#const n = ...` statements can be used to define constants.
- The `#hide/#show` statements can be used to influence which literals are shown in the solution.
AnsProlog: Choice functions

- The literal \{b_1; \ldots ; b_m\}
is true iff any subset of the set \{b_1, \ldots , b_m\} is true.

**Example**

Generate all interpretations over the atoms \(a(1), a(2), a(3)\):

\{ a(1); a(2); a(3) \}.

With **strong negation**:

\(-a(X) :- \text{not } a(X), X=1..3.\)

\{ a(1..3) \}.
AnsProlog: Choice with cardinality

The literal \( l \{ b_1; \ldots; b_m \} u \)

is true iff at least \( l \) and at most \( u \) atoms (included) are true

within the set \( \{ b_1, \ldots, b_m \} \).

Example

Generate all interpretations over the atoms \( a(1), a(2), a(3), b(1), b(2) \)

that contain exactly 2 true atoms:

\[
2 \{ a(1..3); b(1..2) \} 2.
\]

Generate all interpretations over the atoms \( a(1), a(2), a(3), b(1), b(2), b(3) \)

that do not contain exactly 2 or more true atoms for the same

predicate:

\[
\{ a(1..3); b(1..3) \}.
\]

\[
:- 2 \{ a(1..3) \} 3.
\]

\[
:- 2 \{ b(1..3) \} 3.
\]
AnsProlog: Domains of variables

- The domain of a variable must be known in order to avoid “unsafe”-error while the program is grounded.
- The domain can be set literal-wise, rule-wise, or program wise.
- For limiting the scope within a literal use the syntax:
  \[ a(X) : \text{dom}(X) \text{ or } a(X) : X=1..3 \]

Example

\[
\begin{align*}
\text{num}(0..10). \\
\text{even}(2*X) & : \text{num}(X), 2*X \leq 10. \\
1 & \{ a(X) : \text{even}(X) \} 1. \\
\#\text{show } a/1.
\end{align*}
\]
Example: Graph coloring

Example

#const n = 2.
c(1..n).
1 {color(X,I) : c(I)} 1 :- v(X).
:- color(X,I), color(Y,I), e(X,Y), c(I).

% Instance
v(1..4).
e(1,2).
e(1,3).
e(2,4).
e(3,4).
% e(2,3).

#show color/2.
Generate and test

ASP programs are often organized in a “generate-and-test” style: first describe candidate solutions, then rule out possible solutions by stating constraints.

Example

% n-Queens encoding %
#const n = 4.

% Generate possible positions %
1 { q(I,1..n) } 1 :- I = 1..n.

% Rule out attacking positions %
:- q(I1,J), q(I2,J), I1 != I2.
:- q(I,J), q(I+D,J+D), D = 1..n.
:- q(I,J), q(I+D,J-D), D = 1..n.
Generate and test: Further example

**Problem:** In a graph find cliques of size $\geq n$

**Example**

```prolog
#const n = 3.

edge(X,Y) :- edge(Y,X).
n {clique(X) : node(X)}.
:- clique(X), clique(Y), node(X), node(Y), X!=Y, not edge(X,Y).

% Instance %
node(1..5).
edge(1,2;4).
edge(2,3;4).
edge(3,4).
edge(4,2;5).

#show clique/1.
```
The language is even bigger than that! It includes

- Disjunction in the head
- Other operators: #sum,#min,#max,#even,#odd,#avg, ...
- Multi-criteria optimizations
- Heuristic optimizations
- ...

(More on that in the exercises!)
Literature

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