Principles of Knowledge Representation and Reasoning
Nonmonotonic Reasoning

Bernhard Nebel, Felix Lindner, and Thorsten Engesser
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Introduction
A reasoning task

- *If Mary has an essay to write, she will study late in the library.*
- *She has an essay to write.*
A reasoning task

- If Mary has an essay to write, she will study late in the library.
- She has an essay to write.

What do you conclude?
A reasoning task

- *If Mary has an essay to write, she will study late in the library.*
- *She has an essay to write.*

In empirical studies 95% of all subjects conclude (modus ponens):

- *She will study late in the library.*
A reasoning task

- If Mary has an essay to write, she will study late in the library.
- If the library is open, she will study late in the library.
- She has an essay to write.

What do you conclude now?
A reasoning task

- If Mary has an essay to write, she will study late in the library.
- If the library is open, she will study late in the library.
- She has an essay to write.

In cognitive studies now only 60% of the subjects conclude:

- She will study late in the library.
A reasoning task

- *If Mary has an essay to write, she will study late in the library.*
- *She has an essay to write.*

Conclusion?
- *She will study late in the library.*

Reasoning tasks like this (suppression task; Byrne, 1989) suggest that humans often do not reason as suggested by classical logics.
Nonmonotonic reasoning

How can we deal with the reasoning task given in the example? We can use a different representation that allows to restate the task as follows:

- *If Mary has an essay to write, she usually will study late in the library.*
- *She has an essay to write.*
- *If the library is not open, she will not study late in the library.*
- ...
All logics presented so far are monotonic.
Nonmonotonic reasoning

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- A logic is called **monotonic** if all (logical) conclusions from a knowledge base remain justified when new information is added to the knowledge base.
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- When humans reason they use:
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- Cognitive studies indicate that everyday reasoning is often nonmonotonic (Stenning & Lambalgen, 2008; Johnson-Laird, 2010, etc.).
- When humans reason they use:
  - rules that may have exceptions:
    - *If Mary has an essay to write, she normally will study late in the library.*
Nonmonotonic reasoning

- All logics presented so far are monotonic.
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- Cognitive studies indicate that everyday reasoning is often nonmonotonic (Stenning & Lambalgen, 2008; Johnson-Laird, 2010, etc.).
- When humans reason they use:
  - rules that may have **exceptions**:
    
    *If Mary has an essay to write, she normally will study late in the library.*
  - **default** assumptions:
    
    *The library is open.*
Defaults in knowledge bases

Often we use default assumptions when definite information is not available or when we want to fix a standard value:

1. employee(anne)
2. employee(bert)
3. employee(carla)
4. employee(detlef)
5. employee(thomas)
6. onUnpaidMPaternityLeave(thomas)
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7. employee(X) ∧ ¬ onUnpaidMPaternityLeave(X) → gettingSalary(X)
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7. employee(X) ∧ ¬ onUnpaidMPaternityLeave(X) → gettingSalary(X)
8. Typically: employee(X) → ¬ onUnpaidMPaternityLeave(X)
Defaults in common sense reasoning

1. **Tweety** is a bird like other birds.
2. During the summer he stays in **Northern Europe**, in the winter he stays in **Africa**.
Defaults in common sense reasoning

1. **Tweety** is a *bird* like other birds.
2. During the summer he stays in *Northern Europe*, in the winter he stays in *Africa*.

- Would you expect Tweety to be able to fly?
- How does Tweety get from Northern Europe to Africa?
Defaults in common sense reasoning

1. **Tweety** is a *bird* like other birds.
2. During the summer he stays in **Northern Europe**, in the winter he stays in **Africa**.

   - Would you expect Tweety to be able to fly?
   - How does Tweety get from Northern Europe to Africa?

How would you formalize this in *formal logic* so that you get the expected answers?
A formalization ...

1. bird(tweety)
2. spend-summer(tweety, northern-europe) \land spend-winter(tweety, africa)
3. \forall x (bird(x) \rightarrow can-fly(x))
4. far-away(northern-europe, africa)
5. \forall xyz (can-fly(x) \land far-away(y, z) \land spend-summer(x, y) \land spend-winter(x, z) \rightarrow flies(x, y, z))
A formalization …

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- **But:** The implication (3) is just a reasonable assumption.
- What if Tweety is an emu?
Examples of such reasoning patterns

**Closed world assumption:** Database of ground atoms. All ground atoms not present are assumed to be false.

**Negation as failure:** In PROLOG, NOT(P) means “P is not provable” instead of “P is provably false”.

**Non-strict inheritance:** An attribute value is inherited only if there is no more specialized information contradicting the attribute value.

**Reasoning about actions:** When reasoning about actions, it is usually assumed that a property changes only if it has to change, i.e., properties by default do not change.
Default, defeasible, and nonmonotonic reasoning

Default reasoning: Jump to a conclusion if there is no information that contradicts the conclusion.

Defeasible reasoning: Reasoning based on assumptions that can turn out to be wrong: conclusions are defeasible. In particular, default reasoning is defeasible.

Nonmonotonic reasoning: In classical logic, the set of consequences grows monotonically with the set of premises. If reasoning is defeasible, then reasoning becomes nonmonotonic.
Approaches to nonmonotonic reasoning

- **Consistency-based:** Extend classical theory by rules that test whether an assumption is consistent with existing beliefs

  ⇒ Nonmonotonic logics such as DL (default logic), NMLP (nonmonotonic logic programming)
Approaches to nonmonotonic reasoning

- **Consistency-based:** Extend classical theory by rules that test whether an assumption is consistent with existing beliefs
  - Nonmonotonic logics such as DL (default logic), NMLP (nonmonotonic logic programming)

- **Entailment-based on normal models:** Models are ordered by normality. Entailment is determined by considering the most normal models only.
  - Circumscription, preferential and cumulative logics
NM Logic – Consistency-based

If $\phi$ typically implies $\psi$, $\phi$ is given, and it is consistent to assume $\psi$, then conclude $\psi$. 
NM Logic – Consistency-based

If $\varphi$ typically implies $\psi$, $\varphi$ is given, and it is consistent to assume $\psi$, then conclude $\psi$.

1. Typically bird($x$) implies can-fly($x$)
2. $\forall x (\text{emu}(x) \rightarrow \text{bird}(x))$
3. $\forall x (\text{emu}(x) \rightarrow \neg \text{can-fly}(x))$
4. bird(tweety)

$\Rightarrow$ can-fly(tweety)
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4. bird(tweety)

$\Rightarrow$ can-fly(tweety)

5. $\ldots + \text{emu}(tweety)$

$\Rightarrow \neg \text{can-fly}(tweety)$
NM Logic – Normal models

If $\varphi$ typically implies $\psi$, then the models satisfying $\varphi \land \psi$ should be more normal than those satisfying $\varphi \land \neg \psi$. 

1. $\forall x (\text{bird}(x) \land \neg \text{Ab}(x) \rightarrow \text{can-fly}(x))$
2. $\forall x (\text{emu}(x) \rightarrow \text{bird}(x))$
3. $\forall x (\text{emu}(x) \rightarrow \neg \text{can-fly}(x))$
4. $\text{bird}(\text{tweety})$
5. $\text{emu}(\text{tweety})$

Now in all models (incl. the normal ones): $\neg \text{can-fly}(\text{tweety})$
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**Similar idea:** try to minimize the interpretation of “Abnormality” predicates.

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4. \( \text{bird(tweety)} \)

Minimize interpretation of \( \text{Ab} \):
\[ \Rightarrow \text{can-fly(tweety)} \]
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Minimize interpretation of \( \text{Ab} \):

\[ \Rightarrow \text{can-fly(tweety)} \]

5. \( \ldots + \text{emu(tweety)} \)

\[ \Rightarrow \text{Now in all models (incl. the normal ones): } \neg \text{can-fly(tweety)} \]
Default Logic
Default Logic – Outline

Introduction

Default Logic
   Basics
   Extensions
   Properties of extensions
   Normal defaults
   Default proofs
   Decidability

Complexity of Default Logic

Special Kinds of Defaults
Reiter’s default logic: motivation

- We want to express something like “typically birds fly”.
- Add non-logical inference rule

\[
\text{bird}(x) : \text{can-fly}(x) \\
\overline{\text{can-fly}(x)}
\]

with the intended meaning:

*If x is a bird and if it is consistent to assume that x can fly, then conclude that x can fly.*
Reiter’s default logic: motivation

- We want to express something like “typically birds fly”.
- Add non-logical inference rule

\[ \text{bird}(x) : \text{can-fly}(x) \]
\[ \frac{\text{can-fly}(x)}{\text{can-fly}(x)} \]

with the intended meaning:

*If x is a bird and if it is consistent to assume that x can fly, then conclude that x can fly.*

- Exceptions can be represented as formulae:

\[ \forall x (\text{penguin}(x) \rightarrow \neg \text{can-fly}(x)) \]
\[ \forall x (\text{emu}(x) \rightarrow \neg \text{can-fly}(x)) \]
\[ \forall x (\text{kiwi}(x) \rightarrow \neg \text{can-fly}(x)) \]
FOL with classical provability relation $\vdash$ and deductive closure: $\text{Th}(\Phi) := \{ \phi | \Phi \vdash \phi \}$
Formal framework

- **FOL** with classical provability relation $\vdash$ and deductive closure: $\text{Th}(\Phi) := \{ \phi \mid \Phi \vdash \phi \}$

- **Default rules:**
  \[
  \frac{\alpha \cdot \beta}{\gamma}
  \]
  
  - $\alpha$: **Prerequisite**: must have been derived before rule can be applied.
  - $\beta$: **Consistency condition**: the negation may not be derivable.
  - $\gamma$: **Consequence**: will be concluded.

- A default rule is **closed** if it does not contain free variables.

- **(Closed) default theory**: A pair $\langle D, W \rangle$, where $D$ is a countable set of (closed) default rules and $W$ is a countable set of FOL formulae.
Extensions of default theories

Default theories extend the theory given by $W$ using the default rules in $D$ ($\leadsto$ extensions). There may be zero, one, or many extensions.

Example

$W = \{a, \neg b \lor \neg c\}$

$D = \{a : b, b, a : c, c\}$

One extension contains $b$, the other contains $c$. Intuitively, an extension is a set of beliefs resulting from $W$ and $D$. 

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One extension contains $b$, the other contains $c$.

Intuitively, an extension is a set of beliefs resulting from $W$ and $D$. 
Decision problems about extensions in default logic

Existence of extensions: Does a default theory have an extension?
Existence of extensions: Does a default theory have an extension?

Credulous reasoning: If $\varphi$ is in at least one extension, $\varphi$ is a credulous default conclusion.

Skeptical reasoning: If $\varphi$ is in all extensions, $\varphi$ is a skeptical default conclusion.
Extensions (informally)

Desirable properties of an extension $E$ of $\langle D, W \rangle$:

1. Contains all facts: $W \subseteq E$.
2. Is deductively closed: $E = \text{Th}(E)$.
3. All applicable default rules have been applied:
   
   If
   
   1. $(\alpha:\beta) \in D$,
   2. $\alpha \in E$,
   3. $\neg \beta \notin E$
   
   then $\gamma \in E$. 

Further requirement: Application of default rules must follow in sequence (groundedness).
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   then $\gamma \in E$.

- Further requirement: Application of default rules must follow in sequence (groundedness).
Example

\[ W = \emptyset \]

\[ D = \left\{ \frac{a: b}{b}, \frac{b: a}{a} \right\} \]

**Question:** Should \( \text{Th}(\{a, b\}) \) be an extension?
Groundedness

Example

\[ W = \emptyset \]
\[ D = \left\{ \frac{a \colon b}{b}, \frac{b \colon a}{a} \right\} \]

*Question:* Should \( \text{Th}\{a, b\} \) be an extension?

*Answer:* No!

\( a \) can only be derived if we already have derived \( b \).

\( b \) can only be derived if we already have derived \( a \).
Extensions (formally)

**Definition**

Let $\Delta = \langle D, W \rangle$ be a closed default theory. Let $E$ be any set of closed formulae. Define:

$$E_0 = W$$

$$E_i = \text{Th}(E_{i-1}) \cup \left\{ \gamma \bigg| \frac{\alpha: \beta}{\gamma} \in D, \alpha \in E_{i-1}, \neg \beta \notin E \right\}$$
Extensions (formally)

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E_i = \text{Th}(E_{i-1}) \cup \left\{ \gamma \left| \frac{\alpha: \beta}{\gamma} \in D, \alpha \in E_{i-1}, \neg \beta \notin E \right. \right\}
$$

$E$ is called an extension of $\Delta$ if

$$
E = \bigcup_{i=0}^{\infty} E_i.
$$
How to use this definition?

The definition does not tell us how to construct an extension.

However, it tells us how to check whether a set is an extension:

1. Guess a set $E$.
2. Then construct sets $E_i$ by starting with $W$.
3. If $E = \bigcup_{i=0}^{\infty} E_i$, then $E$ is an extension of $\langle D, W \rangle$. 

Examples

\[ D = \left\{ \frac{a: b}{b}, \frac{b: a}{a} \right\} \quad W = \{a \lor b\} \]

\[ D = \left\{ \frac{a: b}{\neg b} \right\} \quad W = \emptyset \]

\[ D = \left\{ \frac{a: b}{\neg b} \right\} \quad W = \{a\} \]

\[ D = \left\{ \frac{a: b : c}{a, b, c} \right\} \quad W = \{b \rightarrow \neg a \land \neg c\} \]

\[ D = \left\{ \frac{c: d : e}{\neg d', \neg e', \neg f} \right\} \quad W = \emptyset \]

\[ D = \left\{ \frac{c: d}{\neg d', \neg c} \right\} \quad W = \emptyset \]

\[ D = \left\{ \frac{a: b}{c}, \frac{a: d}{e} \right\} \quad W = \{a, \neg b \lor \neg d\} \]
Questions, questions, questions . . .

- What can we say about the existence of extensions?
- How are the different extensions related to each other?
  - Can one extension be a subset of another one?
  - Are extensions pairwise incompatible (i.e. jointly inconsistent)?
- Can an extension be inconsistent?
Properties of extensions: existence

Theorem

1. If $W$ is inconsistent, there is only one extension.
2. A closed default theory $\langle D, W \rangle$ has an inconsistent extensions $E$ if and only if $W$ is inconsistent.
Properties of extensions: existence

Theorem

1. If \( W \) is inconsistent, there is only one extension.
2. A closed default theory \( \langle D, W \rangle \) has an inconsistent extensions \( E \) if and only if \( W \) is inconsistent.

Proof idea.

1. If \( W \) is inconsistent, no default rule is applicable and \( \text{Th}(W) \) is the only extension (which is inconsistent as well).
2. Claim 1 \( \iff \) the if-part.
   For only if: Let \( W \) be consistent and assume that there exists an inconsistent extension \( E \).
   Then there exists a consistent \( E_i \) such that \( E_{i+1} \) is inconsistent.
   That is, there is at least one applied default \( \alpha_i : \beta_i / \gamma_i \) with \( \gamma_i \in E_{i+1} \setminus \text{Th}(E_i) \), \( \alpha_i \in E_i \), and \( \neg \beta_i \notin E \).
   But this contradicts the inconsistency of \( E \).
Properties of extensions

Theorem

If $E$ and $F$ are extensions of $\langle D, W \rangle$ such that $E \subseteq F$, then $E = F$. 

Proof sketch.

$E = \bigcup_{i=0}^{\infty} E_i$ and $F = \bigcup_{i=0}^{\infty} F_i$. Use induction to show $F_i \subseteq E_i$.

Base case $i = 0$: Trivially $E_0 = F_0 = W$.

Inductive case $i \geq 1$: Assume $\gamma \in F_{i+1}$. Two cases:

1. $\gamma \in \text{Th}(F_i)$ implies $\gamma \in \text{Th}(E_i)$ (because $F_i \subseteq E_i$ by IH), and therefore $\gamma \in E_{i+1}$.

2. Otherwise $\alpha: \beta \in D$, $\alpha \in F_i$, $\neg \beta \notin F_i$. However, then we have $\alpha \in E_i$ (because $F_i \subseteq E_i$) and $\neg \beta \notin E_i$ (because of $E \subseteq F$), i.e., $\gamma \in E_{i+1}$.
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2. Otherwise $\frac{\alpha: \beta \gamma}{\gamma} \in D$, $\alpha \in F_i$, $\neg \beta \notin F$. However, then we have $\alpha \in E_i$ (because $F_i \subseteq E_i$) and $\neg \beta \notin E$ (because of $E \subseteq F$), i.e., $\gamma \in E_{i+1}$.

\[\square\]
Normal default theories

All defaults in a normal default theory are normal:

\[
\frac{\alpha : \beta}{\beta}.
\]

Theorem
Normal default theories have at least one extension.

Proof sketch.
If \( W \) inconsistent, trivial.
Otherwise construct \( E_0 = W \)

\[ E_{i+1} = \text{Th}(E_i) \cup T_i \]

where \( T_i \) is a maximal set s.t. (1) \( E_i \cup T_i \) is consistent and (2) if \( \beta \in T_i \) then there is \( \alpha : \beta \beta \in D \) and \( \alpha \in E_i \).

Show:
\[ T_i = \{ \beta | \alpha \vdash \beta \beta \in D, \alpha \in E_i, \neg \beta \not\in E_i \} \]

for all \( i \geq 0 \).
Normal default theories

All defaults in a **normal default theory** are **normal**:

\[
\begin{align*}
\alpha : \beta \\
\beta
\end{align*}
\]

**Theorem**

*Normal default theories have at least one extension.*
Normal default theories

All defaults in a normal default theory are normal:

\[ \frac{\alpha : \beta}{\beta} \]

**Theorem**

*Normal default theories have at least one extension.*

**Proof sketch.**

If \( W \) inconsistent, trivial.
Otherwise construct

\[
\begin{align*}
E_0 &= W \\
E_{i+1} &= \text{Th}(E_i) \cup T_i \\
E &= \bigcup_{i=0}^\infty E_i
\end{align*}
\]

where \( T_i \) is a maximal set s.t. (1) \( E_i \cup T_i \) is consistent and (2) if \( \beta \in T_i \) then there is \( \frac{\alpha : \beta}{\beta} \in D \) and \( \alpha \in E_i \).
### Theorem (Orthogonality)

Let $E$ and $F$ be distinct extensions of a normal default theory. Then $E \cup F$ is inconsistent.

### Proof.

Let $E = \bigcup E_i$ and $F = \bigcup F_i$
Normal default theories: extensions are orthogonal

**Theorem (Orthogonality)**

Let $E$ and $F$ be distinct extensions of a normal default theory. Then $E \cup F$ is inconsistent.

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Let $E = \bigcup E_i$ and $F = \bigcup F_i$ with

$$E_{i+1} = \text{Th}(E_i) \cup \left\{ \beta \mid \frac{\alpha : \beta}{\beta} \in D, \alpha \in E_i, \neg \beta \notin E \right\}$$
Normal default theories: extensions are orthogonal

**Theorem (Orthogonality)**

*Let E and F be distinct extensions of a normal default theory. Then E ∪ F is inconsistent.*

**Proof.**

Let $E = \bigcup E_i$ and $F = \bigcup F_i$ with

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and the same for $F$.

Since $E \neq F$, there exists a smallest $i$ such that $E_{i+1} \neq F_{i+1}$. 
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and the same for $F$.

Since $E \neq F$, there exists a smallest $i$ such that $E_{i+1} \neq F_{i+1}$. This means there exists $\alpha : \frac{\beta}{\beta} \in D$ with $\alpha \in E_i = F_i$, but with, say, $\beta \in E_{i+1}$ and $\beta \not\in F_{i+1}$. 
Normal default theories: extensions are orthogonal

**Theorem (Orthogonality)**

Let $E$ and $F$ be distinct extensions of a normal default theory. Then $E \cup F$ is inconsistent.

**Proof.**

Let $E = \bigcup E_i$ and $F = \bigcup F_i$ with

$$E_{i+1} = \text{Th}(E_i) \cup \left\{ \beta \mid \frac{\alpha : \beta}{\beta} \in D, \alpha \in E_i, \neg \beta \not\in E \right\}$$

and the same for $F$.

Since $E \neq F$, there exists a smallest $i$ such that $E_{i+1} \neq F_{i+1}$. This means there exists $\frac{\alpha : \beta}{\beta} \in D$ with $\alpha \in E_i = F_i$, but with, say, $\beta \in E_{i+1}$ and $\beta \not\in F_{i+1}$. This is only possible if $\neg \beta \in F$.

This means, $\beta \in E$ and $\neg \beta \in F$, i.e., $E \cup F$ is inconsistent.
Default proofs in normal default theories

**Definition**

A default proof of $\gamma$ in a normal default theory $\langle D, W \rangle$ is a finite sequence of defaults $(\delta_i = \frac{\alpha_i}{\beta_i})_{i=1,...,n}$ in $D$ such that

1. $W \cup \{\beta_1, \ldots, \beta_n\} \vdash \gamma$,
2. $W \cup \{\beta_1, \ldots, \beta_n\}$ is consistent, and
3. $W \cup \{\beta_1, \ldots, \beta_k\} \vdash \alpha_{k+1}$, for $0 \leq k \leq n - 1$. 

**Theorem**

Let $\Delta = \langle D, W \rangle$ be a normal default theory so that $W$ is consistent. Then $\gamma$ has a default proof in $\Delta$ if and only if there exists an extension $E$ of $\Delta$ such that $\gamma \in E$. 

Test 2 (consistency) in the proof procedure suggests that default provability is not even semi-decidable.
Default proofs in normal default theories

**Definition**

A default proof of \( \gamma \) in a normal default theory \( \langle D, W \rangle \) is a finite sequence of defaults \( (\delta_i = \frac{\alpha_i}{\beta_i})_{i=1,\ldots,n} \) in \( D \) such that

1. \( W \cup \{\beta_1, \ldots, \beta_n\} \models \gamma \),
2. \( W \cup \{\beta_1, \ldots, \beta_n\} \) is consistent, and
3. \( W \cup \{\beta_1, \ldots, \beta_k\} \models \alpha_{k+1} \), for \( 0 \leq k \leq n - 1 \).

**Theorem**

Let \( \Delta = \langle D, W \rangle \) be a normal default theory so that \( W \) is consistent. Then \( \gamma \) has a default proof in \( \Delta \) if and only if there exists an extension \( E \) of \( \Delta \) such that \( \gamma \in E \).

Test 2 (consistency) in the proof procedure suggests that default provability is not even semi-decidable.
Decidability

Theorem

It is not semi-decidable to test whether a formula follows (skeptically or credulously) from a default theory.

Proof.

Let \( \langle D, W \rangle \) be a default theory with \( W = \emptyset \) and \( D = \{ \frac{\beta}{\beta} \} \) with \( \beta \) an arbitrary closed FOL formula. Clearly, \( \beta \) is in some/all extensions of \( \langle D, W \rangle \) if and only if \( \beta \) is satisfiable.

The existence of a semi-decision procedure for default proofs implies that there is a semi-decision procedure for satisfiability in FOL. But this is not possible because FOL validity is semi-decidable and this together with semi-decidability of FOL satisfiability would imply decidability of FOL, which is not the case.
Complexity of Default Logic
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**Propositional default logic**

- **Propositional DL** is decidable.
- How difficult is reasoning in propositional DL?
Propositional default logic

- Propositional DL is decidable.
- How difficult is reasoning in propositional DL?
- The skeptical default reasoning problem (does $\varphi$ follow from $\Delta$ skeptically: $\Delta \models \varphi$?) is called PDS, credulous reasoning is called LPDS.
Propositional default logic

- **Propositional DL** is decidable.
- How difficult is reasoning in propositional DL?
- The **skeptical default reasoning** problem
  (does \( \varphi \) follow from \( \Delta \) skeptically: \( \Delta \models \sim \varphi \)) is called **PDS**, credulous reasoning is called **LPDS**.

- PDS is **coNP-hard**:
  consider \( D = \emptyset, W = \emptyset \)

- LPDS is **NP-hard**:
  consider \( D = \left\{ \frac{\beta}{\beta} \right\}, W = \emptyset \).
Skeptical reasoning in propositional DL

Lemma

\[ PDS \in \Pi^p_2. \]
Skeptical reasoning in propositional DL

**Lemma**

\[ \text{PDS} \in \Pi_2^p. \]

**Proof sketch.**

We show that the complementary problem UNPDS (is there an extension \( E \) such that \( \varphi \notin E \)) is in \( \Sigma_2^p \).

The algorithm:

1. **Guess** set \( T \subseteq D \) of defaults, those that are applied.
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1. **Guess** set \( T \subseteq D \) of defaults, those that are applied.
2. **Verify** that defaults in \( T \) lead to \( E \), using a SAT oracle and the guessed \( E := \text{Th} \left( \left\{ \gamma : \frac{\alpha : \beta}{\gamma} \in T \right\} \cup W \right) \).

\[ \Rightarrow \text{UNPDS} \in \Sigma_2^p. \]
Skeptical reasoning in propositional DL

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3. **Verify** that \( \left\{ \gamma : \frac{\alpha : \beta}{\gamma} \in T \right\} \cup W \not\vdash \varphi \) (SAT oracle).
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\[ \rightsquigarrow \text{UNPDS} \in \Sigma^p_2. \]
Lemma

\( PDS \) is \( \Pi^p_2 \)-hard.

Proof sketch.

Reduction from 2QBF to UNPDS:
Lemma

\[ \text{PDS is } \Pi^p_2\text{-hard.} \]

Proof sketch.

Reduction from 2QBF to UNPDS: For \( \exists \vec{a} \forall \vec{b} \, \varphi(\vec{a}, \vec{b}) \) with \( \vec{a} = a_1, \ldots, a_n \) and \( \vec{b} = b_1, \ldots, b_m \) construct \( \Delta = \langle D, W \rangle \) with

\[
D = \left\{ \frac{a_i}{a_i}, \frac{\neg a_i}{\neg a_i}, \frac{\varphi(\vec{a}, \vec{b})}{\varphi(\vec{a}, \vec{b})} \right\}, \quad W = \emptyset
\]

No extension contains both \( a_i \) and \( \neg a_i \).
Lemma

PDS is $\Pi_2^p$-hard.

Proof sketch.

Reduction from 2QBF to UNPDS: For $\exists \vec{a} \forall \vec{b} \, \varphi(\vec{a}, \vec{b})$ with $\vec{a} = a_1, \ldots, a_n$ and $\vec{b} = b_1, \ldots, b_m$ construct $\Delta = \langle D, W \rangle$ with

$$D = \left\{ \frac{a_i}{a_i}, \frac{-a_i}{-a_i}, \frac{\varphi(\vec{a}, \vec{b})}{\varphi(\vec{a}, \vec{b})} \right\}, \quad W = \emptyset$$

No extension contains both $a_i$ and $-a_i$. Then:
**Lemma**

*PDS is $\Pi_2^p$-hard.*

**Proof sketch.**

Reduction from 2QBF to UNPDS: For $\exists \vec{a} \land \vec{b} \varphi(\vec{a}, \vec{b})$ with $\vec{a} = a_1, \ldots, a_n$ and $\vec{b} = b_1, \ldots, b_m$ construct $\Delta = \langle D, W \rangle$ with

$$D = \left\{ \begin{array}{ccc}
: a_i & : -a_i & : \varphi(\vec{a}, \vec{b}) \\
\frac{a_i}{a_i} & \frac{-a_i}{-a_i} & \frac{\varphi(\vec{a}, \vec{b})}{\varphi(\vec{a}, \vec{b})} \\
\end{array} \right\}, \quad W = \emptyset$$

No extension contains both $a_i$ and $-a_i$. Then:

$\Delta \not\vdash \neg \varphi(\vec{a}, \vec{b})$
Lemma

PDS is $\Pi_2^p$-hard.

Proof sketch.

Reduction from 2QBF to UNPDS: For $\exists \vec{a} \forall \vec{b} \varphi(\vec{a}, \vec{b})$ with $\vec{a} = a_1, \ldots, a_n$ and $\vec{b} = b_1, \ldots, b_m$ construct $\Delta = \langle D, W \rangle$ with

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No extension contains both $a_i$ and $\neg a_i$. Then:

$\Delta \not\models \neg \varphi(\vec{a}, \vec{b})$ iff there is an extension $E$ s.t. $\neg \varphi(\vec{a}, \vec{b}) \not\in E$
Lemma

\( PDS \) is \( \Pi_2^p \)-hard.

Proof sketch.

Reduction from 2QBF to UNPDS: For \( \exists \vec{a} \forall \vec{b} \, \phi(\vec{a}, \vec{b}) \) with \( \vec{a} = a_1, \ldots, a_n \) and \( \vec{b} = b_1, \ldots, b_m \) construct \( \Delta = \langle D, W \rangle \) with

\[
D = \left\{ \frac{a_i}{a_i}, \frac{-a_i}{-a_i}, \frac{\phi(\vec{a}, \vec{b})}{\phi(\vec{a}, \vec{b})} \right\}, \quad W = \emptyset
\]

No extension contains both \( a_i \) and \( -a_i \). Then:

\( \Delta \not\models \neg \phi(\vec{a}, \vec{b}) \) iff there is an extension \( E \) s.t. \( \neg \phi(\vec{a}, \vec{b}) \notin E \)

iff there is \( E \) s.t. \( \phi(\vec{a}, \vec{b}) \in E \) (by \( \frac{\phi(\vec{a}, \vec{b})}{\phi(\vec{a}, \vec{b})} \in D \))
Lemma

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Reduction from 2QBF to UNPDS: For \( \exists \vec{a} \forall \vec{b} \varphi(\vec{a}, \vec{b}) \) with \( \vec{a} = a_1, \ldots, a_n \) and \( \vec{b} = b_1, \ldots, b_m \) construct \( \Delta = \langle D, W \rangle \) with

\[
D = \left\{ \begin{array}{c}
\frac{a_i}{a_i}, \quad \frac{\neg a_i}{\neg a_i}, \quad \frac{\varphi(\vec{a}, \vec{b})}{\varphi(\vec{a}, \vec{b})}
\end{array} \right\}, \quad W = \emptyset
\]

No extension contains both \( a_i \) and \( \neg a_i \). Then:

\( \Delta \not\models \neg \varphi(\vec{a}, \vec{b}) \) iff there is an extension \( E \) s.t. \( \neg \varphi(\vec{a}, \vec{b}) \notin E \)

iff there is \( E \) s.t. \( \varphi(\vec{a}, \vec{b}) \in E \) (by \( \varphi(\vec{a}, \vec{b}) \in D \))

iff there is \( A \subseteq \{a_1, \neg a_1, \ldots, a_n, \neg a_n\} \) s.t. \( A \models \varphi(\vec{a}, \vec{b}) \)
Lemma

\( PDS \) is \( \Pi^p_2 \)-hard.

Proof sketch.

Reduction from 2QBF to UNPDS: For \( \exists \overrightarrow{a} \forall \overrightarrow{b} \varphi(\overrightarrow{a}, \overrightarrow{b}) \) with \( \overrightarrow{a} = a_1, \ldots, a_n \) and \( \overrightarrow{b} = b_1, \ldots, b_m \) construct \( \Delta = \langle D, W \rangle \) with

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No extension contains both \( a_i \) and \( \neg a_i \). Then:

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iff there is \( A \subseteq \{a_1, \neg a_1, \ldots, a_n, \neg a_n\} \) s.t. \( A \models \varphi(\overrightarrow{a}, \overrightarrow{b}) \)

iff \( \exists \overrightarrow{a} \forall \overrightarrow{b} \varphi(\overrightarrow{a}, \overrightarrow{b}) \) is true.
Conclusions & remarks

Theorem

\[ \text{PDS is } \Pi^p_2\text{-complete, even for defaults of the form } \frac{\alpha}{\alpha}. \]

Theorem

\[ \text{LPDS is } \Sigma^p_2\text{-complete, even for defaults of the form } \frac{\alpha}{\alpha}. \]
Conclusions & remarks

**Theorem**

\[ PDS \text{ is } \Pi^p_2\text{-complete, even for defaults of the form } \vdash \alpha / \alpha. \]

**Theorem**

\[ LPDS \text{ is } \Sigma^p_2\text{-complete, even for defaults of the form } \vdash \alpha / \alpha. \]

- PDS is “easier” than reasoning in most modal logics.
- General and normal defaults have the same complexity.
- Polynomial special cases cannot be achieved by restricting, for example, to Horn clauses (satisfiability testing in polynomial time).
- It is necessary to restrict the underlying monotonic reasoning problem and the number of extensions.
- Similar results hold for other nonmonotonic logics.
Special Kinds of Defaults
Semi-normal defaults are sometimes useful:

\[
\frac{\alpha : \beta \land \gamma}{\beta}
\]
Semi-normal defaults (1)

**Semi-normal** defaults are sometimes useful:

\[
\alpha : \beta \land \gamma \\
\hline
\beta
\]

Important when one has **interacting** defaults:

Adult(\(x\)) : Employed(\(x\))

Employed(\(x\))

Student(\(x\)) : Adult(\(x\))

\(\neg\) Employed(\(x\))

For \(\text{Student(TOM)}\) we get two extensions: one with \(\text{Employed(TOM)}\) and the other one with \(\neg\text{Employed(TOM)}\).

Since the third rule is "more specific", we may prefer it.
Semi-normal defaults (1)

Semi-normal defaults are sometimes useful:

\[ \alpha : \beta \land \gamma \]

\[ \beta \]

Important when one has interacting defaults:

\[
\begin{align*}
\text{Adult}(x) : & \quad \text{Employed}(x) \\
\rightarrow & \quad \text{Employed}(x) \\
\text{Student}(x) : & \quad \text{Adult}(x) \\
\rightarrow & \quad \text{Adult}(x) \\
\text{Student}(x) : & \quad \neg \text{Employed}(x) \\
\rightarrow & \quad \neg \text{Employed}(x)
\end{align*}
\]
Semi-normal defaults (1)

Semi-normal defaults are sometimes useful:

\[
\begin{align*}
\alpha & : \beta \land \gamma \\
\hline
\beta
\end{align*}
\]

Important when one has interacting defaults:

\[
\begin{align*}
\text{Adult}(x): & \quad \text{Employed}(x) \\
\hline
\text{Employed}(x)
\end{align*}
\]

\[
\begin{align*}
\text{Student}(x): & \quad \text{Adult}(x) \\
\hline
\text{Adult}(x)
\end{align*}
\]

\[
\begin{align*}
\text{Student}(x): & \quad \neg\text{Employed}(x) \\
\hline
\neg\text{Employed}(x)
\end{align*}
\]

For \text{Student}(TOM) we get two extensions: one with \text{Employed}(TOM) and the other one with \neg\text{Employed}(TOM).
Semi-normal defaults (1)

Semi-normal defaults are sometimes useful:

$$\alpha : \beta \land \gamma$$

$$\frac{\beta}{\beta}$$

Important when one has interacting defaults:

$$\text{Adult}(x) : \text{Employed}(x)$$

$$\text{Employed}(x)$$

$$\text{Student}(x) : \text{Adult}(x)$$

$$\text{Adult}(x)$$

$$\text{Student}(x) : \neg\text{Employed}(x)$$

$$\neg\text{Employed}(x)$$

For \text{Student}(TOM) we get two extensions: one with \text{Employed}(TOM) and the other one with \neg\text{Employed}(TOM). Since the third rule is “more specific”, we may prefer it.
Since being a student is an exception, we could use a \textit{semi-normal} default to exclude students from employed adults:
Semi-normal defaults (2)

Since being a student is an exception, we could use a semi-normal default to exclude students from employed adults:

\[
\text{Student}(x): \quad \neg \text{Employed}(x) \\
\quad \neg \text{Employed}(x) \\
\text{Adult}(x): \quad \text{Employed}(x) \land \neg \text{Student}(x) \\
\quad \text{Employed}(x) \\
\text{Student}(x): \quad \text{Adult}(x) \\
\quad \text{Adult}(x)
\]

Representing conflict-resolution by semi-normal defaults becomes clumsy when the number of default rules becomes high. A scheme for assigning priorities would be more elegant (there are indeed such schemes).
Semi-normal defaults (2)

Since being a student is an exception, we could use a semi-normal default to exclude students from employed adults:

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\text{Student}(x) : & \quad \neg \text{Employed}(x) \\
& \quad \neg \text{Employed}(x) \\
\text{Adult}(x) : & \quad \text{Employed}(x) \land \neg \text{Student}(x) \\
& \quad \text{Employed}(x) \\
\text{Student}(x) : & \quad \text{Adult}(x) \\
& \quad \text{Adult}(x)
\end{align*}
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Semi-normal defaults (2)

- Since being a student is an exception, we could use a semi-normal default to exclude students from employed adults:

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  \begin{align*}
  \text{Student}(x) : & \quad \neg \text{Employed}(x) \\
  & \quad \neg \text{Employed}(x) \\
  \text{Adult}(x) : & \quad \text{Employed}(x) \land \neg \text{Student}(x) \\
  & \quad \text{Employed}(x) \\
  \text{Student}(x) : & \quad \text{Adult}(x) \\
  & \quad \text{Adult}(x)
  \end{align*}
  \]

- Representing conflict-resolution by semi-normal defaults becomes clumsy when the number of default rules becomes high.

- A scheme for assigning priorities would be more elegant (there are indeed such schemes).
Our examples included open defaults, but the theory covers only closed defaults.

If we have $\frac{\alpha(\bar{x})\beta(\bar{x})}{\gamma(\bar{x})}$, then the variables should stand for all nameable objects.

**Problem**: What about objects that have been introduced implicitly, e.g., via formulae such as $\exists x P(x)$.

**Solution by Reiter**: Skolemization of all formulae in $W$ and $D$.

**Interpretation**: An open default stands for all the closed defaults resulting from substituting ground terms for the variables.
Open defaults (2)

Skolemization can create problems because it preserves satisfiability, but it is not an equivalence transformation.
Open defaults (2)

Skolemization can create problems because it preserves satisfiability, but it is not an equivalence transformation.

Example

\[ \forall x (\text{Man}(x) \leftrightarrow \neg \text{Woman}(x)) \]
\[ \forall x (\text{Man}(x) \rightarrow (\exists y (\text{Spouse}(x, y) \land \text{Woman}(y)) \lor \text{Bachelor}(x))) \]
\text{Man}(\text{TOM})
\text{Spouse}(\text{TOM}, \text{MARY})
\text{Woman}(\text{MARY})
\hline
\text{Man}(\text{MARY})
\begin{align*}
\text{Man}(x) \\
\text{Man}(x)
\end{align*}

Skolemization of \( \exists y: \ldots \) enables concluding \text{Bachelor}(\text{TOM})!
The reason is that for \( g(\text{TOM}) \) we get \( \text{Man}(g(\text{TOM})) \) \text{by default} (where \( g \) is the Skolem function).
Open defaults (3)

It is even worse: Logically equivalent theories can have different extensions:
Open defaults (3)

It is even worse: Logically equivalent theories can have different extensions:

\[ W_1 = \{ \exists x (P(c,x) \lor Q(c,x)) \} \]
\[ W_2 = \{ \exists x P(c,x) \lor \exists x Q(c,x) \} \]
\[ D = \left\{ \frac{P(x,y) \lor Q(x,y): R}{R} \right\} \]
Open defaults (3)

It is even worse: Logically equivalent theories can have different extensions:

\[ W_1 = \{ \exists x (P(c,x) \lor Q(c,x)) \} \]
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\[ D = \left\{ \frac{P(x,y) \lor Q(x,y)}{R} : R \right\} \]

\( W_1 \) and \( W_2 \) are logically equivalent. However, the Skolemization of \( W_1 \), symbolically \( s(W_1) \), is not equivalent with \( s(W_2) \). The only extension of \( \langle D, W_1 \rangle \) is \( \text{Th}(s(W_1) \cup R) \). The only extension of \( \langle D, W_2 \rangle \) is \( \text{Th}(s(W_2)) \).
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It is even worse: Logically equivalent theories can have different extensions:

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\( W_1 \) and \( W_2 \) are logically equivalent. However, the Skolemization of \( W_1 \), symbolically \( s(W_1) \), is not equivalent with \( s(W_2) \). The only extension of \( \langle D, W_1 \rangle \) is \( \text{Th}(s(W_1) \cup R) \). The only extension of \( \langle D, W_2 \rangle \) is \( \text{Th}(s(W_2)) \).

**Note**: Skolemization is not the right method to deal with open defaults in the general case.
Although Reiter’s definition of DL makes sense, one can come up with a number of variations and extend the investigation …

- Extensions can be defined differently (e.g., by remembering consistency conditions).
- … or by removing the groundedness condition.
- Open defaults can be handled differently (more model-theoretically).
- General proof methods for the finite, decidable case
- Applications of default logic:
  - Diagnosis
  - Reasoning about actions
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