Principles of Knowledge Representation and Reasoning

Semantic Networks and Description Logics V: Description Logics – Decidability and Complexity

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Decidability & Undecidability

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Subsumption and satisfiability of concept descriptions is decidable in description logics using only the following concept and role forming operators: $C \sqcap D$, $C \sqcup D$, $\neg C$, $\forall r.C$, $\exists r.C$, $r \sqsubseteq s$, $r \sqcap s$, $r \sqcup s$, $\neg r$, r^{-1}.

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Potential problems: Role composition and cardinality restrictions for role fillers. Cardinality restrictions are not a real problem.

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- $r \circ s$, $r \doteq s$, $C \sqcap D$, $\forall r.C$ [Schmidt-Schauß 89]

  … This is, in fact, a fragment of the early description logic KL-ONE, where people had hoped to come up with a complete subsumption algorithm

Polynomial Cases

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$$C := A \mid \neg A \mid \top \mid \bot \mid C \sqcap C' \mid \forall r.C \mid (\geq n\, r) \mid (\leq n\, r),$$
$$r := t \mid r^{-1}$$

and

$$C := A \mid C \sqcap C' \mid \forall r.C \mid \exists r$$
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Complexity of $\mathcal{ALC}$ Subsumption

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Unsatisfiability and subsumption are reducible to each other. We give a reduction from UNSAT. A propositional formula φ over the atoms a_i is mapped to $\pi(\varphi)$:

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\begin{aligned}
a_i &\mapsto a_i \\
\psi \wedge \psi' &\mapsto \pi(\psi) \sqcap \pi(\psi') \\
\psi' \vee \psi &\mapsto \pi(\psi) \sqcup \pi(\psi') \\
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Obviously, φ is satisfiable iff $\pi(\varphi)$ is satisfiable (use structural induction).

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- Unlikely – since models of a single concept description can already become exponentially large!
- We will show PSPACE-completeness, whereby hardness is proved using a complexity result for (un)satisifiability in the modal logic K.
- Satisifiability and unsatisfiability in K is PSPACE-complete.

Reduction from K-satisfiability

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Theorem (Complexity of $\mathcal{ALC}$)

$\mathcal{ALC}$ subsumption, unsatisfiability and satisfiability are all PSPACE-complete.

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$\rightsquigarrow$ DL can be thought as fragments of first-order predicate logic. However, they are much more similar to modal logics.

$\rightsquigarrow$ Algorithms and complexity results can be borrowed. Works also the other way around.

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Expressive Power vs. Complexity

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- Of course, one wants to have a description logic with high expressive power. However, high expressive power implies usually that the computational complexity of the reasoning problems might also be high, e.g., $\mathcal{FL}^-$ vs. $\mathcal{ALC}$.

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  3. Use expressive description logics with complete inference algorithms.
- For a long time, only options 1 and 2 were studied. Meanwhile, most researcher concentrate on option 3!

The Complexity of Subsumption in TBoxes

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- Unfolding C_n leads to a concept description with a size $\Omega(2^n)$.
- Is it possible to avoid this blowup? Can we avoid exponential preprocessing?

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- Unfolding + strucural subsumption gives us an exponential algorithm.

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Proof sketch.

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$$\begin{aligned} \text{automaton } A &\mapsto \text{terminology } \mathcal{T}_A \\ \text{state } q &\mapsto \text{concept name } q \\ \text{terminal state } q_f &\mapsto \text{concept name } q_f \\ \text{input symbol } r &\mapsto \text{role name } r \end{aligned}$$

r-transition from q to $q' \mapsto q \doteq \ldots \sqcap \forall r : q' \sqcap \ldots$. $\square$

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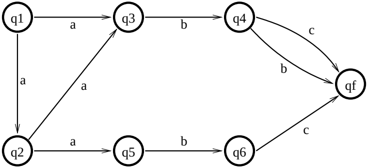
“Proof” by example



$$q_1 \doteq \forall a.q_3 \sqcap \forall a.q_2$$

$$q_2 \doteq \forall a.q_3 \sqcap \forall a.q_5$$

$$q_3 \doteq \forall b.q_4$$

$$q_4 \doteq \forall b.q_f \sqcap \forall c.q_f$$

$$q_5 \doteq \forall b.q_6$$

$$q_6 \doteq \forall b.q_f$$

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$$q_2 \equiv \forall abb.q_f \sqcap \forall abc.q_f$$

$$q_1 \sqsubseteq_{\mathcal{T}} q_2 \quad \text{and} \quad \mathcal{L}(q_2) \subseteq \mathcal{L}(q_1)$$

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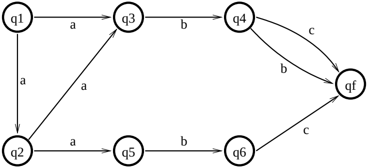

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$$q_1 \sqsubseteq_{\mathcal{T}} q_2 \quad \text{and} \quad \mathcal{L}(q_2) \subseteq \mathcal{L}(q_1)$$

In general, we have: $\mathcal{L}(q) \subseteq \mathcal{L}(q')$ iff $q' \sqsubseteq_{\mathcal{T}} q$

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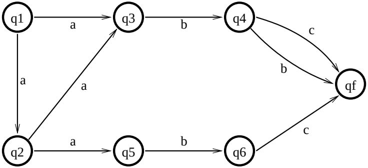

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$q_1 \sqsubseteq_{\mathcal{T}} q_2$ and $\mathcal{L}(q_2) \subseteq \mathcal{L}(q_1)$

In general, we have: $\mathcal{L}(q) \subseteq \mathcal{L}(q')$ iff $q' \sqsubseteq_{\mathcal{T}} q$, from which the correctness of the reduction and the complexity result follows.

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- Pathological situations do not happen very often.
- In fact, if the definition depth is logarithmic in the size of the TBox, the whole problem vanishes.
- However, in order to protect oneself against such problems, one often uses lazy unfolding …

What does this complexity result mean?

- Note that for expressive languages such as $\mathcal{ALC}$, we do not notice any difference!
- The TBox subsumption complexity result for less expressive languages does not play a large role in practice
- Pathological situations do not happen very often.
- In fact, if the definition depth is logarithmic in the size of the TBox, the whole problem vanishes.
- However, in order to protect oneself against such problems, one often uses lazy unfolding . . .
- Similarly, also for $\mathcal{ALC}$ concept descriptions, one notices that they are usually very well behaved.

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- Nowadays tools can handle KBs with up to 160,000 concepts (example from unified medical language system) in reasonable time.
- Description logics are used as the semantic backbone for OWL (a Web-language extending RDF).

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