Decidability & Undecidability

Polynomial Cases
Complexity of $\mathcal{ALC}$ Subsumption
Expressive Power vs. Complexity
The Complexity of Subsumption in TBoxes
Outlook
Literature
Decidability

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$L_2^=$: $L_2$ plus equality.
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$L_2^=$ is decidable.

**Corollary**

Subsumption and satisfiability of concept descriptions is decidable in description logics using only the following concept and role forming operators: $C \sqcap D$, $C \sqcup D$, $\neg C$, $\forall r.C$, $\exists r.C$, $r \sqsubseteq s$, $r \sqcap s$, $r \sqcup s$, $\neg r$, $r^{-1}$. 
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**Potential problems:** Role composition and cardinality restrictions for role fillers. Cardinality restrictions are not a real problem.
Undecidability

- $r \circ s$, $r \sqcap s$, $\neg r$, 1 [Schild 88]

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- $r \circ s$, $r \cdot s = s$, $C \cap D$, $\forall r.C$ [Schmidt-Schauß 89]
  
  ... This is, in fact, a fragment of the early description logic KL-ONE, where people had hoped to come up with a complete subsumption algorithm
Polynomial Cases
Decidable, polynomial-time cases

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$$ C := A \mid \neg A \mid \top \mid \bot \mid C \sqcap C' \mid \forall r. C \mid (\geq nr) \mid (\leq nr), $$
  $$ r := t \mid r^{-1} $$

and

$$ C := A \mid C \sqcap C' \mid \forall r. C \mid \exists r $$
  $$ r := t \mid r^{-1} \mid r \sqcap r' \mid r \circ r' $$
Complexity of $\textit{ALC}$ Subsumption
How hard is $\mathcal{ALC}$ subsumption?

**Proposition**

$\mathcal{ALC}$ subsumption and unsatisfiability are co-NP-hard.

**Proof.**

Unsatisfiability and subsumption are reducible to each other. We give a reduction from UNSAT.

A propositional formula $\phi$ over the atoms $a_i$ is mapped to $\pi(\phi)$:

- $a_i \mapsto \top$  
- $\psi \land \psi' \mapsto \pi(\psi) \land \pi(\psi')$  
- $\psi \lor \psi' \mapsto \pi(\psi) \lor \pi(\psi')$  
- $\neg \psi \mapsto \neg \pi(\psi)$

Obviously, $\phi$ is satisfiable iff $\pi(\phi)$ is satisfiable (use structural induction).

If $\phi$ has a model, construct a model for $\pi(\phi)$ with just one element $t$ standing for the truth of the atoms and the formula. Conversely, if $\pi(\phi)$ satisfiable, pick one element $d \in \pi(\phi)_I$ and set the truth value of atom $a_i$ according to the fact that $d \in \pi(a_i)_I$.
How hard is \( \text{ALC} \) subsumption?

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\begin{align*}
a_i & \mapsto a_i \\
\psi \land \psi' & \mapsto \pi(\psi) \land \pi(\psi') \\
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\[\Box\]
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- Unlikely – since models of a single concept description can already become exponentially large!
- We will show PSPACE-completeness, whereby hardness is proved using a complexity result for (un)satisfiability in the modal logic $\mathcal{K}$.
- Satisfiability and unsatisfiability in $\mathcal{K}$ is PSPACE-complete.
Reduction from $K$-satisfiability

**Lemma (Lower bound for $\mathcal{ALC}$)**

$\mathcal{ALC}$ subsumption, unsatisfiability and satisfiability are all PSPACE-hard.
Reduction from $K$-satisfiability

Lemma (Lower bound for $\mathcal{ALC}$)

$\mathcal{ALC}$ subsumption, unsatisfiability and satisfiability are all $PSPACE$-hard.

Proof.

Extend the reduction given in the last proof by the following two rules — assuming that $b$ is a fixed role name:

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Again, obviously, $\phi$ is satisfiable iff $\pi(\phi)$ is satisfiable (again using structural induction). If $\phi$ has a Kripke model, interpret each world $w$ as an object in the universe of discourse, that is, $w$ is an instance of the primitive concept $\pi(a_i)$ iff $a_i$ is true in $w$. 

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Lemma (Upper Bound for $\mathcal{ALC}$)

$\mathcal{ALC}$ subsumption, unsatisfiability and satisfiability are all in $\text{PSPACE}$. 

Proof.

This follows from the tableau algorithm for $\mathcal{ALC}$. Although there may be exponentially many closed constraint systems, we can visit them step by step generating only one at a time. When closing a system, we have to consider only one role at a time – resulting in an only polynomial space requirement, i.e., satisfiability can be decided in $\text{PSPACE}$.

Theorem (Complexity of $\mathcal{ALC}$)

$\mathcal{ALC}$ subsumption, unsatisfiability and satisfiability are all $\text{PSPACE}$-complete.


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Further consequences of the reducibility of $K$ to $ALC$

In the reduction we used only one role symbol. Are there modal logics that would require more than one such role symbol?
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  - The multi-modal logic $K_n$ has $n$ different Box operators $\Box_i$ (for $n$ different agents).
  - $ALC$ (wrt. TBox reasoning) is a notational variant of $K_n$. [Schild, IJCAI-91]
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- Are there other modal logics that correspond to other descriptions logics?
  - $\Rightarrow$ propositional dynamic logic (PDL), e.g., transitive closure, composition, role inverse, ...

- $\Rightarrow$ DL can be thought as fragments of first-order predicate logic. However, they are much more similar to modal logics.
- $\Rightarrow$ Algorithms and complexity results can be borrowed. Works also the other way around.
Expressive Power vs. Complexity
Of course, one wants to have a description logic with high expressive power. However, high expressive power implies usually that the computational complexity of the reasoning problems might also be high, e.g., $\mathcal{FL}^-$ vs. $\mathcal{ALC}$. 

There are three approaches to this problem:

1. Use only small description logics with complete inference algorithms.
2. Use expressive description logics, but employ incomplete inference algorithms.
3. Use expressive description logics with complete inference algorithms.

For a long time, only options 1 and 2 were studied. Meanwhile, most researchers concentrate on option 3!
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- In the following example unfolding leads to an exponential blowup:

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  C_1 = \forall r.C_0 \sqcap \forall s.C_0
  \]
  \[
  C_2 = \forall r.C_1 \sqcap \forall s.C_1
  \]
  \[
  \vdots
  \]
  \[
  C_n = \forall r.C_{n-1} \sqcap \forall s.C_{n-1}
  \]

  Unfolding \(C_n\) leads to a concept description with a size \(\Omega(2^n)\). Is it possible to avoid this blowup? Can we avoid exponential preprocessing?
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Question: Can we decide in polynomial time TBox subsumption for a description logic such as $\mathcal{FL}^-$, for which concept subsumption in the empty TBox can be decided in polynomial time?
TBox subsumption for small languages

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Unfolding + structural subsumption gives us an exponential algorithm.
Theorem (Complexity of TBox subsumption)

*TBox subsumption for $\mathcal{FL}_0$ is NP-hard.*

Proof sketch. We use the NDFA-equivalence problem, which is NP-complete for cycle-free automatons and PSPACE-complete for general NDFA.

We transform a cycle-free NDFA to a $\mathcal{FL}_0$-terminology with the mapping $\pi$ as follows:

- Automaton $A$ maps to terminology $T_A$
- State $q$ maps to concept name $q$
- Terminal state $q_f$ maps to concept name $q_f$
- Input symbol $r$ maps to role name $r$
- $r$-transition from $q$ to $q'$ maps to $\exists r : q' \sqsubseteq \ldots$
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$r$-transition from $q$ to $q' \mapsto q = \ldots \sqcap \forall r : q' \sqcap \ldots$
“Proof” by example

In general, we have:

$L(q) \subseteq L(q')$ iff $q' \sqsubseteq T q$, from which the correctness of the reduction and the complexity result follows.
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q_3 & \equiv \forall b.q_4 \\
q_4 & \equiv \forall b.q_f \sqcap \forall c.q_f \\
q_5 & \equiv \forall b.q_6 \\
q_6 & \equiv \forall b.q_f \\
q_1 & \sqsubseteq_T q_2 \text{ and } \mathcal{L}(q_2) \subseteq \mathcal{L}(q_1)
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“Proof” by example

In general, we have: \( \mathcal{L}(q) \subseteq \mathcal{L}(q') \) iff \( q' \sqsubseteq_T q \), from which the correctness of the reduction and the complexity result follows.
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- However, in order to protect oneself against such problems, one often uses lazy unfolding …
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- Pathological situations do not happen very often.
- In fact, if the definition depth is logarithmic in the size of the TBox, the whole problem vanishes.
- However, in order to protect oneself against such problems, one often uses lazy unfolding …
- Similarly, also for $\mathcal{ALC}$ concept descriptions, one notices that they are usually very well behaved.
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- Nowadays tools can handle KBs with up to 160,000 concepts (example from unified medical language system) in reasonable time.
- Description logics are used as the semantic backbone for OWL (a Web-language extending RDF).
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