Principles of Knowledge Representation and Reasoning
Semantic Networks and Description Logics IV: Description Logics – Algorithms

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Motivation
Reasoning problems & algorithms

Reasoning problems:

- **Satisfiability** or **subsumption** of concept descriptions
Reasoning problems & algorithms

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- Satisfiability or subsumption of concept descriptions
- Satisfiability or instance relation in ABoxes
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Solving techniques presented in this chapter:

- **Structural subsumption algorithms**
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- Structural subsumption algorithms
  - Normalization of concept descriptions and structural comparison
  - very fast, but can only be used for small DLs
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Reasoning problems:
- **Satisfiability** or **subsumption** of concept descriptions
- **Satisfiability** or **instance relation** in ABoxes

Solving techniques presented in this chapter:
- **Structural subsumption algorithms**
  - **Normalization** of concept descriptions and **structural comparison**
  - very fast, but can only be used for small DLs
- **Tableau algorithms**
  - Similar to modal tableau methods
  - Often the method of choice
Structural Subsumption Algorithms
Structural subsumption algorithms

In what follows we consider the rather small logic $\mathcal{FL}^-$:

- $C \sqcap D$
- $\forall r. C$
- $\exists r$ (simple existential quantification)
Structural subsumption algorithms

In what follows we consider the rather small logic $\mathcal{FL}^-$:

- $C \sqcap D$
- $\forall r.C$
- $\exists r$ (simple existential quantification)

To solve the subsumption problem for this logic we apply the following idea:

1. In the conjunction, collect all universally quantified expressions (also called value restrictions) with the same role and build complex value restriction:

   $$\forall r.C \sqcap \forall r.D \rightarrow \forall r.(C \sqcap D).$$

2. Compare all conjuncts with each other. For each conjunct in the subsuming concept there should be a corresponding one in the subsumed one.
Example

\[ D = \text{Human} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child}. \text{Human} \sqcap \]
\[ \forall \text{has-child.} \exists \text{has-child} \]
\[ C = \text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child} \sqcap \]
\[ \forall \text{has-child.}(\text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child}) \]
Example

\[ D = \text{Human} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child.} \text{Human} \sqcap \forall \text{has-child.} \exists \text{has-child} \]
\[ C = \text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child.}(\text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child}) \]

Check: \( C \subseteq D \)

1. Collect value restrictions in \( D \):
   
   \[ ... \forall \text{has-child.}(\text{Human} \sqcap \exists \text{has-child}) \]
Example

\[ D = \text{Human} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child}. \text{Human} \sqcap \] 
\[ \forall \text{has-child}. \exists \text{has-child} \]
\[ C = \text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child} \sqcap \] 
\[ \forall \text{has-child}. (\text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child}) \]

Check: \( C \sqsubseteq D \)

1. **Collect** value restrictions in \( D \):
   \[ \ldots \forall \text{has-child}. (\text{Human} \sqcap \exists \text{has-child}) \]
2. **Compare**:
   1. For \( \text{Human} \) in \( D \), we have \( \text{Human} \) in \( C \).
   2. For \( \exists \text{has-child} \) in \( D \), we have \( \exists \text{has-child} \) in \( C \).
   3. For \( \forall \text{has-child}. (...) \) in \( D \), we have \( \text{Human} \) and \( \exists \text{has-child} \) in \( C \).
Example

\[ D = \text{Human} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child} \cdot \text{Human} \sqcap \forall \text{has-child} \cdot \exists \text{has-child} \]
\[ C = \text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child} \cdot (\text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child}) \]

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1. **Collect** value restrictions in \( D \):
   ...
   \( \forall \text{has-child} \cdot (\text{Human} \sqcap \exists \text{has-child}) \)

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   1. For \( \text{Human} \) in \( D \), we have \( \text{Human} \) in \( C \).
   2. For \( \exists \text{has-child} \) in \( D \), we have \( \exists \text{has-child} \) in \( C \).
   3. For \( \forall \text{has-child} \cdot (\ldots) \) in \( D \), we have \( \text{Human} \) and \( \exists \text{has-child} \) in \( C \).

\( \implies C \) is subsumed by \( D \)!
Subsumption algorithm

**SUB(C, D) algorithm:**

1. Reorder terms (using **commutativity**, **associativity** and **value restriction law**):

   \[
   C = \bigcap A_i \cap \bigcap \exists r_j \cap \bigcap \forall r_k : C_k
   \]

   \[
   D = \bigcap B_l \cap \bigcap \exists s_m \cap \bigcap \forall s_n : D_n
   \]
Subsumption algorithm

\textbf{SUB}(C, D) algorithm:

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2. For each $B_l$ in $D$, is there an $A_i$ in $C$ with $A_i = B_l$?

\[ \Rightarrow C \sqsubseteq D \text{ iff all questions are answered positively.} \]
Subsumption algorithm

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**Subsumption algorithm**

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2. For each \( B_l \) in \( D \), is there an \( A_i \) in \( C \) with \( A_i = B_l \)?

3. For each \( \exists s_m \) in \( D \), is there an \( \exists r_j \) in \( C \) with \( s_m = r_j \)?

4. For each \( \forall s_n : D_n \) in \( D \), is there a \( \forall r_k : C_k \) in \( C \) such that \( s_n = r_k \) and \( C_k \sqsubseteq D_n \) (i.e., check \( \text{SUB}(C_k, D_n) \))?
Subsumption algorithm

**SUB***(\(C, D\)) algorithm:

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2. For each \(B_l\) in \(D\), is there an \(A_i\) in \(C\) with \(A_i = B_l\)?
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4. For each \(\forall s_n : D_n\) in \(D\), is there a \(\forall r_k : C_k\) in \(C\) such that \(s_n = r_k\) and \(C_k \sqsubseteq D_n\) (i.e., check \(\text{SUB}(C_k, D_n)\))? 

\[\implies C \sqsubseteq D\] if all questions are answered positively.
### Soundness

**Theorem (Soundness)**

\[ \text{SUB}(C,D) \Rightarrow C \sqsubseteq D \]

**Proof sketch.**

Reordering of terms step (1):

1. Commutativity and associativity are trivial
Soundness

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1. Commutativity and associativity are trivial
2. Value restriction law. We show: \((\forall r.(C \sqcap D))^\mathcal{I} = (\forall r.C \sqcap \forall r.D)^\mathcal{I}\)
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   Assume \(d \in (\forall r. (C \sqcap D))^\mathcal{I}\).
   If there is no \(e \in D\) with \((d, e) \in r^\mathcal{I}\) it follows trivially that \(d \in (\forall r. C \sqcap \forall r. D)^\mathcal{I}\).
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   Assume \( d \in (\forall r. (C \sqcap D))^I \).
   
   If there is no \( e \in D \) with \( (d, e) \in r^I \) it follows trivially that \( d \in (\forall r. C \sqcap \forall r. D)^I \).

   If there is an \( e \in D \) with \( (d, e) \in r^I \) it follows \( e \in (C \sqcap D)^I = C^I \cap D^I \).
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   \( d \in (\forall r. C \cap \forall r. D)^\mathcal{I} \).

   If there is an \( e \in D \) with \( (d, e) \in r^\mathcal{I} \) it follows \( e \in (C \cap D)^\mathcal{I} \)
   \( = C^\mathcal{I} \cap D^\mathcal{I} \).

   Since \( e \) is arbitrary, we have \( d \in (\forall r. C)^\mathcal{I} \) and \( d \in (\forall r. D)^\mathcal{I} \).
### Theorem (Soundness)

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   If there is no \( e \in D \) with \((d, e) \in r^I\) it follows trivially that
   \( d \in (\forall r.C \cap \forall r.D)^I \).
   If there is an \( e \in D \) with \((d, e) \in r^I\) it follows
   \( e \in (C \cap D)^I = C^I \cap D^I \).
   Since \( e \) is arbitrary, we have
   \( d \in (\forall r.C)^I \) and \( d \in (\forall r.D)^I \),
   i.e.,
   \[ (\forall r.(C \cap D))^I \subseteq (\forall r.C \cap \forall r.D)^I. \]
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If there is no \(e \in D\) with \((d, e) \in r^I\) it follows trivially that \(d \in (\forall r. C \cap \forall r. D)^I\).

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Since \(e\) is arbitrary, we have \(d \in (\forall r. C)^I\) and \(d \in (\forall r. D)^I\), i.e., \((\forall r. (C \cap D))^I \subseteq (\forall r. C \cap \forall r. D)^I\).

The other direction is similar.
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   If there is no \( e \in D \) with \( (d, e) \in r^\mathcal{I} \) it follows trivially that \( d \in (\forall r. C \cap \forall r. D)^\mathcal{I} \).
   
   If there is an \( e \in D \) with \( (d, e) \in r^\mathcal{I} \) it follows \( e \in (C \cap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I} \).
   
   Since \( e \) is arbitrary, we have \( d \in (\forall r. C)^\mathcal{I} \) and \( d \in (\forall r. D)^\mathcal{I} \), i.e., \( (\forall r. (C \cap D))^\mathcal{I} \subseteq (\forall r. C \cap \forall r. D)^\mathcal{I} \).
   
   The other direction is similar.

Steps (2+3+4): Induction on the nesting depth of \( \forall \)-expressions.
Completeness

Theorem (Completeness)

\[ C \sqsubseteq D \Rightarrow \text{SUB}(C, D). \]
Completeness

Theorem (Completeness)

\[ C \subseteq D \implies \text{SUB}(C, D). \]

Proof idea.

One shows the contrapositive:

\[ \neg \text{SUB}(C, D) \implies C \not\subseteq D \]
Completeness

Theorem (Completeness)

\[ C \sqsubseteq D \Rightarrow SUB(C, D). \]

Proof idea.

One shows the contrapositive:

\[ \neg SUB(C, D) \Rightarrow C \nsubseteq D \]

Idea: If one of the rules leads to a negative answer, we use this to construct an interpretation with a special element \( d \) such that

\[ d \in C^\mathcal{I}, \text{ but } d \notin D^\mathcal{I}. \]
Generalizing the algorithm

Extensions of $\mathcal{FL}^-$ by

- $\neg A$ (atomic negation),
Generalizing the algorithm

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- $\neg A$ (atomic negation),
- $(\leq nr), (\geq nr)$ (cardinality restrictions),
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Extensions of $\mathcal{FL}^-$ by

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- $r \circ s$ (role composition)

do not lead to any problems.
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However: If we use full existential restrictions, then it is very unlikely that we can come up with a simple structural subsumption algorithm – having the same flavor as the one above.
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More precisely: There is (most probably) no algorithm that uses polynomially many reorderings and simplifications and allows for a simple structural comparison.
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More precisely: There is (most probably) no algorithm that uses polynomially many reorderings and simplifications and allows for a simple structural comparison.

Reason: Subsumption for $\mathcal{FL}^- + \exists r.C$ is NP-hard (Nutt).
ABox reasoning

**Idea:** Abstraction + classification
ABox reasoning

*Idea*: Abstraction + classification

- **Complete** ABox by propagating value restrictions to role fillers.
Idea: Abstraction + classification

- Complete ABox by propagating value restrictions to role fillers.
- Compute for each object its most specialized concepts.
ABox reasoning

**Idea:** Abstraction + classification

- **Complete** ABox by propagating value restrictions to role fillers.
- Compute for each object its **most specialized concepts**.
- These can then be handled using the ordinary subsumption algorithm.
Tableau Subsumption Method
Tableau method

Logic $\mathcal{ALC}$:

- $C \sqcap D$
- $C \sqcup D$
- $\neg C$
- $\forall r.C$
- $\exists r.C$
Tableau method

Logic $\mathcal{ALC}$:

- $C \cap D$
- $C \sqcup D$
- $\neg C$
- $\forall r. C$
- $\exists r. C$

Idea: Decide (un-)satisfiability of a concept description $C$ by trying to systematically construct a model for $C$. If that is successful, $C$ is satisfiable. Otherwise, $C$ is unsatisfiable.
Example: Subsumption in a TBox

Example

TBox:

Hermaphrodite = Male \sqcap \text{Female}

\text{Parent-of-sons-and-daughters} = \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female}

\text{Parent-of-hermaphrodite} = \exists \text{has-child}. \text{Hermaphrodite}
Example: Subsumption in a TBox

TBox:

\[
\text{Hermaphrodite} = \text{Male} \sqcap \text{Female}
\]

\[
\text{Parent-of-sons-and-daughters} = \exists \text{has-child.Male} \sqcap \exists \text{has-child.Female}
\]

\[
\text{Parent-of-hermaphrodite} = \exists \text{has-child.Hermaphrodite}
\]

Query:

\[
\text{Parent-of-sons-and-daughters} \sqsubseteq \text{Parent-of-hermaphrodites}
\]
Reductions

1 **Unfolding:**

\[
\exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \\
\sqsubseteq \exists \text{has-child}. (\text{Male} \sqcap \text{Female})
\]
Reductions

1. **Unfolding:**
   \[
   \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \\
   \sqsubseteq \exists \text{has-child}. (\text{Male} \sqcap \text{Female})
   \]

2. **Reduction to unsatisfiability:** Is the concept
   \[
   \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqcap \\
   \neg \exists \text{has-child}. (\text{Male} \sqcap \text{Female})
   \]
   unsatisfiable?
Reductions

1. **Unfolding:**
   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqsubseteq \exists \text{has-child}.(\text{Male} \sqcap \text{Female}) \]

2. **Reduction to unsatisfiability:** Is the concept
   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqcap \neg \exists \text{has-child}.(\text{Male} \sqcap \text{Female}) \]
   unsatisfiable?

3. **Negation normal form** (move negations inside):
   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqcap \forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}) \]
Reductions

1. **Unfolding:**
   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqsubseteq \exists \text{has-child}. (\text{Male} \sqcap \text{Female}) \]

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   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqcap \neg \exists \text{has-child}. (\text{Male} \sqcap \text{Female}) \]
   unsatisfiable?

3. **Negation normal form** (move negations inside):
   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqcap \forall \text{has-child}. (\neg \text{Male} \sqcup \neg \text{Female}) \]

4. **Try to construct a model**
Assumption: There exists an object $x$ in the interpretation of our concept:

$$x \in (\exists \ldots)^I$$
1 Assumption: There exists an object \( x \) in the interpretation of our concept:

\[
x \in (\exists \ldots)^I
\]

2 This implies that \( x \) is in the interpretation of all conjuncts:

\[
x \in (\exists \text{has-child}.\text{Male})^I
\]

\[
x \in (\exists \text{has-child}.\text{Female})^I
\]

\[
x \in (\forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}))^I
\]
Model construction (1)

1 Assumption: There exists an object $x$ in the interpretation of our concept:

$$x \in (\exists \ldots)^I$$

2 This implies that $x$ is in the interpretation of all conjuncts:

$$x \in (\exists \text{has-child}.\text{Male})^I$$
$$x \in (\exists \text{has-child}.\text{Female})^I$$
$$x \in (\forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}))^I$$

3 This implies that there should be objects $y$ and $z$ such that $(x,y) \in \text{has-child}^I$, $(x,z) \in \text{has-child}^I$, $y \in \text{Male}^I$ and $z \in \text{Female}^I$, and ...
Model construction (2)

\[ x \vdash \exists \text{has-child}.\text{Male} \]
\[ x \vdash \exists \text{has-child}.\text{Female} \]
Model construction (3)

\[ x: \exists \text{has-child}.\text{Male} \]
\[ x: \exists \text{has-child}.\text{Female} \]
\[ x: \forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}) \]

```
x: \exists has-child.Male
x: \exists has-child.Female
x: \forall has-child.(\neg Male \sqcup \neg Female)
```
Model construction (4)

\[ x : \exists \text{has-child}.\text{Male} \]
\[ x : \exists \text{has-child}.\text{Female} \]
\[ x : \forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}) \]
\[ y : \neg \text{Male} \]

\[
\begin{array}{c}
\text{Male} \\
(\neg \text{Male} \text{ or } \neg \text{Female}) \\
\neg \text{Male} \quad \text{Contradiction}
\end{array}
\]

\[
\begin{array}{c}
\text{Female} \\
(\neg \text{Male} \text{ or } \neg \text{Female})
\end{array}
\]

\[
\begin{array}{c}
\text{has-child} \\
\downarrow \\
\text{y}
\end{array}
\]

\[
\begin{array}{c}
\text{has-child} \\
\downarrow \\
\text{z}
\end{array}
\]

\[
\begin{array}{c}
\text{z}
\end{array}
\]

Motivation
Structural Subsumption Algorithms
Tableau Subsumption Method
Example
Reductions: Unfolding & Unsatisfiability
Model Construction Equivalences & NNF Constraint Systems Transforming Constraint Systems Invariances Soundness and Completeness Space Complexity ABox Reasoning

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Model construction (5)

\[ x : \exists \text{has-child}.\text{Male} \]
\[ x : \exists \text{has-child}.\text{Female} \]
\[ x : \forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}) \]
\[ y : \neg \text{Female} \]
\[ z : \neg \text{Male} \]
Model construction (5)

\[ x: \exists \text{has-child} .\text{Male} \]
\[ x: \exists \text{has-child} .\text{Female} \]
\[ x: \forall \text{has-child}.(\neg\text{Male} \sqcup \neg\text{Female}) \]
\[ y: \neg\text{Female} \]
\[ z: \neg\text{Male} \]

\[ \implies \text{Model constructed!} \]
Tableau method (1): NNF

We write: $C \equiv D$ iff $C \sqsubseteq D$ and $D \sqsubseteq C$. Now we have the following equivalences:

\[
\neg(C \cap D) \equiv \neg C \cup \neg D \\
\neg(\forall r. C) \equiv \exists r. \neg C \\
\neg\neg C \equiv C
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\neg(C \sqcup D) \equiv \neg C \cap \neg D \\
\neg(\exists r. C) \equiv \forall r. \neg C
\]
Tableau method (1): NNF

We write: \( C \equiv D \) iff \( C \sqsubseteq D \) and \( D \sqsubseteq C \). Now we have the following equivalences:

\[
\begin{align*}
\neg (C \cap D) & \equiv \neg C \cup \neg D \\
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\neg \neg C & \equiv C \\
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These equivalences can be used to move all negations signs to the inside, resulting in concept description where only concept names are negated: negation normal form (NNF).
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Theorem (NNF)

The negation normal form of an $\textit{ALC}$ concept can be computed in polynomial time.
Tableau method (2): Constraint systems

A constraint is a syntactical object of the form:

\[ x : C \quad \text{or} \quad x r y , \]

where \( C \) is a concept description in NNF, \( r \) is a role name, and \( x \) and \( y \) are variable names.
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Let \( \mathcal{I} \) be an interpretation with universe \( \mathcal{D} \). An \( \mathcal{I} \)-assignment \( \alpha \) is a function that maps each variable symbol to an object of the universe \( \mathcal{D} \).
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Let \( \mathcal{I} \) be an interpretation with universe \( \mathcal{D} \). An \( \mathcal{I} \)-assignment \( \alpha \) is a function that maps each variable symbol to an object of the universe \( \mathcal{D} \).

A constraint \( x : C (x r y) \) is satisfied by an \( \mathcal{I} \)-assignment \( \alpha \) if \( \alpha(x) \in C^{\mathcal{I}} \) (resp. \( (\alpha(x), \alpha(y)) \in r^{\mathcal{I}} \)).
Tableau method (3): Constraint systems

Definition

A constraint system $S$ is a finite, non-empty set of constraints. An $\mathcal{I}$-assignment $\alpha$ satisfies $S$ if $\alpha$ satisfies each constraint in $S$. $S$ is satisfiable if there exist $\mathcal{I}$ and $\alpha$ such that $\alpha$ satisfies $S$. 

Theorem

An ALC concept $C$ in NNF is satisfiable if and only if the system \{ $x: C$ \} is satisfiable.
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A **constraint system** $S$ is a finite, non-empty set of constraints. An $\mathcal{I}$-assignment $\alpha$ **satisfies** $S$ if $\alpha$ satisfies each constraint in $S$. $S$ is **satisfiable** if there exist $\mathcal{I}$ and $\alpha$ such that $\alpha$ satisfies $S$.

**Theorem**

An $\mathcal{ALC}$ concept $C$ in NNF is satisfiable if and only if the system \( \{ x : C \} \) is satisfiable.
Tableau method (4): Transforming constraint systems

Transformation rules:

1. \( S \rightarrow \cap \{ x : C_1, x : C_2 \} \cup S \)
   if \((x : C_1 \cap C_2) \in S\) and either \((x : C_1)\) or \((x : C_2)\) or both are not in \(S\).
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   - if \( (x : C_1 \sqcup C_2) \in S \) and neither \( (x : C_1) \in S \) nor \( (x : C_2) \in S \) and \( D = C_1 \) or \( D = C_2 \).
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3. $S \rightarrow \exists \{x r y, y : C\} \cup S$
   if $(x : \exists r. C) \in S$, $y$ is a fresh variable, and there is no $z$ s.t. $(x r z) \in S$ and $(z : C) \in S$.

4. $S \rightarrow \forall \{y : C\} \cup S$
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Notice: Deterministic rules (1,3,4) vs. non-deterministic (2).
Generating rules (3) vs. non-generating (1,2,4).
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Theorem (Invariance)

Let $S$ and $T$ be constraint systems.

1. If $T$ has been generated by applying a deterministic rule to $S$, then $S$ is satisfiable if and only if $T$ is satisfiable.

Theorem (Termination)

Let $C$ be an ALC concept description in NNF. Then there exists no infinite chain of transformations starting from the constraint system $\{ x : C \}$.
Theorem (Invariance)

Let $S$ and $T$ be constraint systems.

1. If $T$ has been generated by applying a deterministic rule to $S$, then $S$ is satisfiable if and only if $T$ is satisfiable.

2. If $T$ has been generated by applying a non-deterministic rule to $S$, then $S$ is satisfiable if $T$ is satisfiable. Furthermore, if a non-deterministic rule can be applied to $S$, then it can be applied such that $S$ is satisfiable if and only if the resulting system $T$ is satisfiable.
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Theorem (Termination)

Let $C$ be an $\mathcal{ALC}$ concept description in NNF. Then there exists no infinite chain of transformations starting from the constraint system $\{x : C\}$.
A constraint system is called **closed** if no transformation rule can be applied.
Tableau method (6): Soundness and completeness

A constraint system is called **closed** if no transformation rule can be applied.

A **clash** is a pair of constraints of the form $x : A$ and $x : \neg A$, where $A$ is a concept name.
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**Theorem (Soundness and Completeness)**

A closed constraint system is satisfiable if and only if it does not contain a clash.
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**Theorem (Soundness and Completeness)**

*A closed constraint system is satisfiable if and only if it does not contain a clash.*

**Proof idea.**

$\Rightarrow$: obvious. $\Leftarrow$: Construct a model by using the concept labels.
Space requirements

Because the tableau method is non-deterministic (→⊔ rule), there could be exponentially many closed constraint systems in the end.
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Interestingly, applying the rules on a single constraint system can lead to constraint systems of exponential size.

Example

\[
\exists r.A \land \exists r.B \\
\forall r. (\exists r.A \land \exists r.B \\
\forall r. (\ldots))
\]

However: One can modify the algorithm so that it needs only polynomial space.

Idea: Generate a \(y\) only for one \(\exists r.C\) and then proceed into the depth.
ABox reasoning

ABox satisfiability can also be decided using the tableau method if we can add constraints of the form $x \neq y$ (for UNA):
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ABox satisfiability can also be decided using the tableau method if we can add constraints of the form $x \neq y$ (for UNA):

- **Normalize** and **unfold** and add inequalities for all pairs of objects mentioned in the ABox.
ABox reasoning

ABox satisfiability can also be decided using the tableau method if we can add constraints of the form $x \not= y$ (for UNA):

- **Normalize** and **unfold** and add inequalities for all pairs of objects mentioned in the ABox.
- Strictly speaking, in $\mathcal{ALC}$ we do not need this because we are never **forced** to identify two objects.
Motivation

Structural Subsumption Algorithms

Tableau Subsumption Method

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