1 Motivation
Reasoning problems & algorithms

Reasoning problems:
- Satisfiability or subsumption of concept descriptions
- Satisfiability or instance relation in ABoxes

Solving techniques presented in this chapter:
- Structural subsumption algorithms
  - Normalization of concept descriptions and structural comparison
  - very fast, but can only be used for small DLs
- Tableau algorithms
  - Similar to modal tableau methods
  - Often the method of choice
2 Structural Subsumption Algorithms

- Idea
- Example
- Algorithm
- Soundness
- Completeness
- Generalizations
- ABox Reasoning
In what follows we consider the rather small logic $\mathcal{FL}^{-}$:

- $C \sqcap D$
- $\forall r. C$
- $\exists r$ (simple existential quantification)

To solve the subsumption problem for this logic we apply the following idea:

1. In the conjunction, collect all universally quantified expressions (also called value restrictions) with the same role and build complex value restriction:

   $$\forall r. C \sqcap \forall r. D \rightarrow \forall r. (C \sqcap D).$$

2. Compare all conjuncts with each other. For each conjunct in the subsuming concept there should be a corresponding one in the subsumed one.
Example

Example

\[ D = \text{Human} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child.Human} \sqcap \forall \text{has-child.} \exists \text{has-child} \]

\[ C = \text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child.}(\text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child}) \]

Check: \( C \sqsubseteq D \)

1. Collect value restrictions in \( D \):
   
   \[ \ldots \forall \text{has-child.}(\text{Human} \sqcap \exists \text{has-child}) \]

2. Compare:
   
   1. For \text{Human} in \( D \), we have \text{Human} in \( C \).
   2. For \( \exists \text{has-child} \) in \( D \), we have \( \exists \text{has-child} \) in \( C \).
   3. For \( \forall \text{has-child.}(\ldots) \) in \( D \), we have \( \text{Human} \) and \( \exists \text{has-child} \) in \( C \).

\[ \implies \text{C is subsumed by } D \]
Subsumption algorithm

**SUB**(\(C, D\)) algorithm:

1. Reorder terms (using **commutativity**, **associativity** and **value restriction law**):

\[
C = \bigcap A_i \cap \bigcap \exists r_j \cap \bigcap \forall r_k : C_k
\]

\[
D = \bigcap B_l \cap \bigcap \exists s_m \cap \bigcap \forall s_n : D_n
\]

2. For each \(B_l\) in \(D\), is there an \(A_i\) in \(C\) with \(A_i = B_l\)?:

3. For each \(\exists s_m\) in \(D\), is there an \(\exists r_j\) in \(C\) with \(s_m = r_j\)?

4. For each \(\forall s_n : D_n\) in \(D\), is there a \(\forall r_k : C_k\) in \(C\) such that \(s_n = r_k\) and \(C_k \sqsubseteq D_n\) (i.e., check \(\text{SUB}(C_k, D_n)\))?  

\[\iff C \sqsubseteq D\] iff all questions are answered positively.
Soundness

**Theorem (Soundness)**

\[ \text{SUB}(C, D) \Rightarrow C \sqsubseteq D \]

**Proof sketch.**

Reordering of terms step (1):

1. Commutativity and associativity are trivial
2. Value restriction law. We show: \( (\forall r.(C \cap D))^I = (\forall r.C \cap \forall r.D)^I \)
   
   Assume \( d \in (\forall r.(C \cap D))^I \).
   
   If there is no \( e \in D \) with \( (d, e) \in r^I \) it follows trivially that \( d \in (\forall r.C \cap \forall r.D)^I \).
   
   If there is an \( e \in D \) with \( (d, e) \in r^I \) it follows \( e \in (C \cap D)^I = C^I \cap D^I \).
   
   Since \( e \) is arbitrary, we have \( d \in (\forall r.C)^I \) and \( d \in (\forall r.D)^I \),
   
   i.e., \( (\forall r.(C \cap D))^I \subseteq (\forall r.C \cap \forall r.D)^I \).

   The other direction is similar.

Steps (2+3+4): Induction on the nesting depth of \( \forall \)-expressions.
Completeness

Theorem (Completeness)

\[ C \sqsubseteq D \Rightarrow SUB(C, D). \]

Proof idea.

One shows the contrapositive:

\[ \neg SUB(C, D) \Rightarrow C \not\sqsubseteq D \]

Idea: If one of the rules leads to a negative answer, we use this to construct an interpretation with a special element \( d \) such that

\[ d \in C^\mathcal{I}, \text{ but } d \not\in D^\mathcal{I}. \]
Generalizing the algorithm

Extensions of $\mathcal{FL}^-$ by

- $\neg A$ (atomic negation),
- $(\leq nr)$, $(\geq nr)$ (cardinality restrictions),
- $r \circ s$ (role composition)

do not lead to any problems.

However: If we use full existential restrictions, then it is very unlikely that we can come up with a simple structural subsumption algorithm – having the same flavor as the one above.

More precisely: There is (most probably) no algorithm that uses polynomially many reorderings and simplifications and allows for a simple structural comparison.

Reason: Subsumption for $\mathcal{FL}^- + \exists r.C$ is NP-hard (Nutt).
ABox reasoning

**Idea**: Abstraction + classification

- **Complete** ABox by propagating value restrictions to role fillers.
- Compute for each object its most specialized concepts.
- These can then be handled using the ordinary subsumption algorithm.
3 Tableau Subsumption Method

- Example
- Reductions: Unfolding & Unsatisfiability
- Model Construction
- Equivalences & NNF
- Constraint Systems
- Transforming Constraint Systems
- Invariances
- Soundness and Completeness
- Space Complexity
- ABox Reasoning
Tableau method

Logic $\mathcal{ALC}$:

- $C \sqcap D$
- $C \sqcup D$
- $\neg C$
- $\forall r.C$
- $\exists r.C$

Idea: Decide (un-)satisfiability of a concept description $C$ by trying to systematically construct a model for $C$. If that is successful, $C$ is satisfiable. Otherwise, $C$ is unsatisfiable.
Example: Subsumption in a TBox

**Example**

**TBox:**

\[
\text{Hermaphrodite} = \text{Male} \sqcap \text{Female} \\
\text{Parent-of-sons-and-daughters} = \\
\exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \\
\text{Parent-of-hermaphrodite} = \exists \text{has-child}. \text{Hermaphrodite}
\]

**Query:**

\[
\text{Parent-of-sons-and-daughters} \sqsubseteq \tau \\
\text{Parent-of-hermaphrodites}
\]
Reductions

1 Unfolding:
\[
\exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \\
\sqsubseteq \exists \text{has-child}. (\text{Male} \sqcap \text{Female})
\]

2 Reduction to unsatisfiability: Is the concept
\[
\exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqcap \\
\neg \exists \text{has-child}. (\text{Male} \sqcap \text{Female})
\]
unsatisfiable?

3 Negation normal form (move negations inside):
\[
\exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqcap \\
\forall \text{has-child}. (\neg \text{Male} \sqcup \neg \text{Female})
\]

4 Try to construct a model
Model construction (1)

1 Assumption: There exists an object $x$ in the interpretation of our concept:

$$x \in (\exists \ldots)^\mathcal{I}$$

2 This implies that $x$ is in the interpretation of all conjuncts:

$$x \in (\exists \text{has-child}. \text{Male})^\mathcal{I}$$
$$x \in (\exists \text{has-child}. \text{Female})^\mathcal{I}$$
$$x \in (\forall \text{has-child}. (\neg \text{Male}) \sqcup (\neg \text{Female}))^\mathcal{I}$$

3 This implies that there should be objects $y$ and $z$ such that $(x, y) \in \text{has-child}^\mathcal{I}, (x, z) \in \text{has-child}^\mathcal{I}, y \in \text{Male}^\mathcal{I}$ and $z \in \text{Female}^\mathcal{I}$, and ...
Model construction (2)

\[ x : \exists \text{has-child}.\text{Male} \]
\[ x : \exists \text{has-child}.\text{Female} \]

Diagram:
- \( x \)
  - \text{has-child} \rightarrow \( y \) (Male)
  - \text{has-child} \rightarrow \( z \) (Female)
Model construction (3)

\[ x : \exists \text{has-child.Male} \]
\[ x : \exists \text{has-child.Female} \]
\[ x : \forall \text{has-child.}(\neg \text{Male} \sqcup \neg \text{Female}) \]

![Diagram](image-url)
Model construction (4)

\[ x : \exists \text{has-child}.\text{Male} \]
\[ x : \exists \text{has-child}.\text{Female} \]
\[ x : \forall \text{has-child}.(\neg\text{Male} \sqcup \neg\text{Female}) \]
\[ y : \neg\text{Male} \]

\[ x \]
\[ \text{has-child} \]
\[ y \]
\[ \text{Male} \]
\[ \neg\text{Male or } \neg\text{Female} \]

\[ z \]
\[ \text{Female} \]
\[ \neg\text{Male or } \neg\text{Female} \]

\[ \neg\text{Male} \rightarrow \text{Contradiction} \]
Model construction (5)

\[ x: \exists \text{has-child}. \text{Male} \]
\[ x: \exists \text{has-child}. \text{Female} \]
\[ x: \forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}) \]
\[ y: \neg \text{Female} \]
\[ z: \neg \text{Male} \]

\[ \rightarrow \text{Model constructed!} \]
Tableau method (1): NNF

We write: $C \equiv D$ iff $C \sqsubseteq D$ and $D \sqsubseteq C$. Now we have the following equivalences:

- $\neg(C \cap D) \equiv \neg C \cup \neg D$
- $\neg(C \cup D) \equiv \neg C \cap \neg D$
- $\neg(\forall r. C) \equiv \exists r. \neg C$
- $\neg(\exists r. C) \equiv \forall r. \neg C$
- $\neg\neg C \equiv C$

These equivalences can be used to move all negations signs to the inside, resulting in concept description where only concept names are negated: negation normal form (NNF).

Theorem (NNF)

The negation normal form of an $\mathcal{ALC}$ concept can be computed in polynomial time.
Tableau method (2): Constraint systems

A constraint is a syntactical object of the form:

\[ x : C \quad \text{or} \quad x r y, \]

where \( C \) is a concept description in NNF, \( r \) is a role name, and \( x \) and \( y \) are variable names.

Let \( \mathcal{I} \) be an interpretation with universe \( \mathcal{D} \). An \( \mathcal{I} \)-assignment \( \alpha \) is a function that maps each variable symbol to an object of the universe \( \mathcal{D} \).

A constraint \( x : C (x r y) \) is satisfied by an \( \mathcal{I} \)-assignment \( \alpha \) if \( \alpha(x) \in C^\mathcal{I} \) (resp. \( (\alpha(x), \alpha(y)) \in r^\mathcal{I} \)).
Tableau method (3): Constraint systems

Definition

A constraint system $S$ is a finite, non-empty set of constraints. An $I$-assignment $\alpha$ satisfies $S$ if $\alpha$ satisfies each constraint in $S$. $S$ is satisfiable if there exist $I$ and $\alpha$ such that $\alpha$ satisfies $S$.

Theorem

An $\mathcal{ALC}$ concept $C$ in NNF is satisfiable if and only if the system $\{x : C\}$ is satisfiable.
Tableau method (4): Transforming constraint systems

Transformation rules:

1. $S \rightarrow \cap \{x : C_1, x : C_2\} \cup S$
   if $(x : C_1 \cap C_2) \in S$ and either $(x : C_1)$ or $(x : C_2)$ or both are not in $S$.

2. $S \rightarrow \sqcup \{x : D\} \cup S$
   if $(x : C_1 \cup C_2) \in S$ and neither $(x : C_1) \in S$ nor $(x : C_2) \in S$ and $D = C_1$ or $D = C_2$.

3. $S \rightarrow \exists \{x r y, y : C\} \cup S$
   if $(x : \exists r.C) \in S$, $y$ is a fresh variable, and there is no $z$ s.t. $(x r z) \in S$ and $(z : C) \in S$.

4. $S \rightarrow \forall \{y : C\} \cup S$
   if $(x : \forall r.C), (x r y) \in S$ and $(y : C) \notin S$.

Notice: Deterministic rules (1,3,4) vs. non-deterministic (2).
Generating rules (3) vs. non-generating (1,2,4).
Tableau method (5): Invariances

Theorem (Invariance)

Let $S$ and $T$ be constraint systems.

1. If $T$ has been generated by applying a deterministic rule to $S$, then $S$ is satisfiable if and only if $T$ is satisfiable.

2. If $T$ has been generated by applying a non-deterministic rule to $S$, then $S$ is satisfiable if $T$ is satisfiable. Furthermore, if a non-deterministic rule can be applied to $S$, then it can be applied such that $S$ is satisfiable if and only if the resulting system $T$ is satisfiable.

Theorem (Termination)

Let $C$ be an $\mathcal{ALC}$ concept description in NNF. Then there exists no infinite chain of transformations starting from the constraint system $\{x : C\}$.
Tableau method (6): Soundness and completeness

A constraint system is called **closed** if no transformation rule can be applied.

A **clash** is a pair of constraints of the form $x : A$ and $x : \neg A$, where $A$ is a concept name.

**Theorem (Soundness and Completeness)**

*A closed constraint system is satisfiable if and only it does not contain a clash.*

**Proof idea.**

$\Rightarrow$: obvious. $\Leftarrow$: Construct a model by using the concept labels.
Space requirements

Because the tableau method is non-deterministic ($\rightarrow\square$ rule), there could be exponentially many closed constraint systems in the end.

Interestingly, applying the rules on a single constraint system can lead to constraint systems of exponential size.

Example

$$\exists r. A \sqcap \exists r. B \sqcap$$
$$\forall r. (\exists r. A \sqcap \exists r. B \sqcap$$
$$\forall r. (\exists r. A \sqcap \exists r. B \sqcap$$
$$\forall r.(\ldots)))$$

However: One can modify the algorithm so that it needs only polynomial space.

Idea: Generate a $y$ only for one $\exists r. C$ and then proceed into the depth.
ABox reasoning

ABox satisfiability can also be decided using the tableau method if we can add constraints of the form $x \neq y$ (for UNA):

- Normalize and unfold and add inequalities for all pairs of objects mentioned in the ABox.
- Strictly speaking, in $\mathcal{ALC}$ we do not need this because we are never forced to identify two objects.
**The Description Logic Handbook: Theory, Implementation, Applications**, 

Hector J. Levesque and Ronald J. Brachman. 
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Manfred Schmidt-Schauß and Gert Smolka. 
Attributive concept descriptions with complements. 

Bernhard Hollunder and Werner Nutt. 
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F. Baader and U. Sattler. 
An Overview of Tableau Algorithms for Description Logics. 

Practical Reasoning for Very Expressive Description Logics. 