Principles of
Knowledge Representation and Reasoning

Semantic Networks and Description Logics III:
Description Logics – Reasoning Services and Reductions

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November 25, 2015

1 Motivation

Example TBox & ABox

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Basic Reasoning Services
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General ABox Reasoning Services
Summary and Outlook

```
     Male ≐ ¬Female
    Human ⊑ Living_entity
    Woman ≐ Human ⊓ Female
      Man ≐ Human ⊓ Male
   Mother ≐ Woman ⊓ ∃has-child.Human
   Father ≐ Man ⊓ ∃has-child.Human
   Parent ≐ Father ⊔ Mother
Grandmother
          ≐ Woman ⊓ ∃has-child.Parent
Mother-without-daughter
          ≐ Mother ⊓ ∀has-child.Male
Mother-with-many-children
          ≐ Mother ⊓ (≥ 3 has-child)
```

```
DIANA:       Woman
ELIZABETH:   Woman
CHARLES:     Man
EDWARD:      Man
ANDREW:      Man
DIANA:       Mother-without-daughter
(ELIZABETH,  CHARLES):  has-child
(ELIZABETH,  EDWARD):   has-child
(ELIZABETH,  ANDREW):   has-child
(DIANA,      WILLIAM):  has-child
(CHARLES,    WILLIAM):  has-child
```


Motivation: Reasoning services

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What do we want to know?

- We want to check whether the knowledge base is reasonable:
  - Is each defined concept in a TBox satisfiable?
  - Is a given TBox satisfiable?
  - Is a given ABox satisfiable?
- What can we conclude from the represented knowledge?
  - Is concept X subsumed by concept Y?
  - Is an object a instance of a concept X?
- These problems can be reduced to logical satisfiability or implication – using the logical semantics.
- *However,* we take a different route: we will try to simplify these problems and then we specify direct inference methods.

2 Basic Reasoning Services

- Satifisfiability without a TBox
- Satisfiability in TBox
- Eliminating the TBox
- Normalization
- Unfolding

Satisfiability of concept descriptions

Satisfiability of concept descriptions

Given a concept description C in "isolation", i.e., in an empty TBox, is C satisfiable?

Test:

- Does there exist an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$?
- Translated into FOL: Is the formula $\exists x\, C(x)$ satisfiable?

Example

$\texttt{Woman} \sqcap (\leq 0\, \texttt{has-child}) \sqcap (\geq 1\, \texttt{has-child})$ is unsatisfiable.

Satisfiability of concept descriptions in a TBox

Satisfiability of concept descriptions in a TBox

Given a TBox $\mathcal{T}$ and a concept description C, is C satisfiable?

Test:

- Does there exist a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \emptyset$?
- Translated into FOL: Is the formula $\exists x\, C(x)$ together with the formulae resulting from the translation of $\mathcal{T}$ satisfiable?

Example

$\texttt{Mother-without-daughter} \sqcap \forall \texttt{has-child.Female}$ is unsatisfiable, given our previously specified family TBox.

Reduction: Getting rid of the TBox

We can reduce satisfiability problem of concept descriptions in a TBox to the satisfiability problem of concept descriptions in the empty TBox.

Idea:

- Since TBoxes are cycle-free, one can understand a concept definition as a kind of "macro".
- For a given TBox $\mathcal{T}$ and a given concept description C, all defined concept symbols appearing in C can be expanded until C contains only undefined concept symbols.
- An expanded concept description is then satisfiable if and only if C is satisfiable in $\mathcal{T}$.
- *Problem*: What do we do with partial definitions (using $\sqsubseteq$)?

Normalized terminologies

- A terminology is called normalized when it does not contain definitions fo the form $A \sqsubseteq C$.
- In order to normalize a terminology, replace

$$A \sqsubseteq C$$

by

$$A \doteq A^* \sqcap C,$$

where A^* is a fresh concept symbol (not appearing elsewhere in $\mathcal{T}$).
- If $\mathcal{T}$ is a terminology, the normalized terminology is denoted by $\widetilde{\mathcal{T}}$.

Normalizing is reasonable

Theorem (Normalization invariance)

If $\mathcal{I}$ is a model of the terminology $\mathcal{T}$, then there exists a model $\mathcal{I}'$ of $\widetilde{\mathcal{T}}$ such that for all concept symbols A occurring in $\mathcal{T}$, it holds $A^{\mathcal{I}} = A^{\mathcal{I}'}$, and vice versa.

Proof.

"$\Rightarrow$": Let $\mathcal{I}$ be a model of $\mathcal{T}$. This model should be extended to $\mathcal{I}'$ so that the freshly introduced concept symbols also get interpretations.
Assume $(A \sqsubseteq C) \in \mathcal{T}$, i.e., we have $(A \doteq A^* \sqcap C) \in \widetilde{\mathcal{T}}$.
Then set $A^{*\mathcal{I}'} := A^{\mathcal{I}}$.
$\mathcal{I}'$ obviously satisfies $\widetilde{\mathcal{T}}$ and has the same interpretation for all symbols in $\mathcal{T}$.
"$\Leftarrow$": Given a model $\mathcal{I}'$ of $\widetilde{\mathcal{T}}$, its restriction to symbols of $\mathcal{T}$ is the interpretation we look for. □

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TBox unfolding

- We say that a normalized TBox is unfolded by one step when all defined concept symbols on the right sides are replaced by their defining terms.
- Example: `Mother` $\doteq$ `Woman` $\sqcap \ldots$ is unfolded to `Mother` $\doteq$ (`Human` $\sqcap$ `Female`) $\sqcap \ldots$
- We write $U(\mathcal{T})$ to denote a one-step unfolding and $U^n(\mathcal{T})$ to denote an n-step unfolding.
- We say that $\mathcal{T}$ is unfolded if $U(\mathcal{T}) = \mathcal{T}$.
- $U^n(\mathcal{T})$ is called the unfolding of $\mathcal{T}$ if $U^n(\mathcal{T}) = U^{n+1}(\mathcal{T})$.
If such an unfolding exists, it is denoted by $\widehat{\mathcal{T}}$.

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Properties of unfoldings (1): Existence

Theorem (Existence of unfolded terminology)

Each normalized terminology $\mathcal{T}$ can be unfolded, i.e., its unfolding $\widehat{\mathcal{T}}$ exists.

Proof idea.

The main reason is that terminologies have to be cycle-free. The proof can be done by induction of the definition depth of concepts. □

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Properties of unfoldings (2): Equivalence

Theorem (Model equivalence for unfolded terminologies)

$\mathcal{I}$ is a model of a normalized terminology $\mathcal{T}$ if and only if it is a model of $\widehat{\mathcal{T}}$.

Proof sketch.

"$\Rightarrow$": Let $\mathcal{I}$ be a model of $\mathcal{T}$.
Then it is also a model of $U(\mathcal{T})$, since on the right side of the definitions only terms with identical interpretations are substituted.
However, then it must also be a model of $\widehat{\mathcal{T}}$.

"$\Leftarrow$": Let $\mathcal{I}$ be a model for $U(\mathcal{T})$. Clearly, this is also a model of $\mathcal{T}$ (with the same argument as above).
This means that any model $\widehat{\mathcal{T}}$ is also a model of $\mathcal{T}$. □

Generating models

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- All concept and role names not occurring on the left hand side of definitions in a terminology $\mathcal{T}$ are called primitive components.
- Interpretations restricted to primitive components are called initial interpretations.

Theorem (Model extension)

For each initial interpretation $\mathcal{J}$ of a normalized TBox, there exists a unique interpretation $\mathcal{I}$ extending $\mathcal{J}$ and satisfying $\mathcal{T}$.

Proof idea.

Use $\widehat{\mathcal{T}}$ and compute an interpretation for all defined symbols. □

Corollary (Model existence for TBoxes)

Each TBox has at least one model

Unfolding of concept descriptions

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- Similar to the unfolding of TBoxes, we can define the unfolding of a concept description.
- We write $\widehat{C}$ for the unfolded version of C.

Theorem (Satisfiability of unfolded concepts)

An concept description C is satisfiable in a terminology $\mathcal{T}$ if and only if $\widehat{C}$ satisfiable in an empty terminology.

Proof.

"$\Rightarrow$": trivial.

"$\Leftarrow$": Use the interpretation for all the symbols in $\widehat{C}$ to generate an initial interpretation of $\mathcal{T}$.
Then extend it to a full model $\mathcal{I}$ of $\mathcal{T}$.
This satisfies $\mathcal{T}$ as well as $\widehat{C}$. Since $\widehat{C}^{\mathcal{I}} = C^{\mathcal{I}}$, it satisfies also C. □

3 General TBox Reasoning Services

- Subsumption
- Subsumption vs. Satisfiability
- Classification

Subsumption in a TBox

Subsumption in a TBox

Given a terminology $\mathcal{T}$ and two concept descriptions C and D, is C subsumed by (or a sub-concept of) D in $\mathcal{T}$ (symb. $C \sqsubseteq_{\mathcal{T}} D$)?

Test:

- Is C interpreted as a subset of D in each model $\mathcal{I}$ of $\mathcal{T}$, i.e. $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$?
- Is the formula $\forall x \big(C(x) \rightarrow D(x)\big)$ a logical consequence of the translation of $\mathcal{T}$ into FOL?

Example

Given our family TBox, it holds $\mathtt{Grandmother} \sqsubseteq_{\mathcal{T}} \mathtt{Mother}$.

Subsumption (without a TBox)

Subsumption (without a TBox)

Given two concept descriptions C and D, is C subsumed by D regardless of a TBox (or in an empty TBox) (symb. $C \sqsubseteq D$)?

Test:

- Is C interpreted as a subset of D for all interpretations $\mathcal{I}$ ($C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$)?
- Is the formula $\forall x \big(C(x) \rightarrow D(x)\big)$ logically valid?

Example

Clearly, $\mathtt{Human} \sqcap \mathtt{Female} \sqsubseteq \mathtt{Human}$.

Reductions

- Subsumption in a TBox can be reduced to subsumption in the empty TBox:
  … normalize and unfold TBox and concept descriptions.
- Subsumption in the empty TBox can be reduced to unsatisfiability:
  … $C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable.
- Unsatisfiability can be reduced to subsumption:
  … C is unsatisfiable iff $C \sqsubseteq (C \sqcap \neg C)$.

Classification

Classification

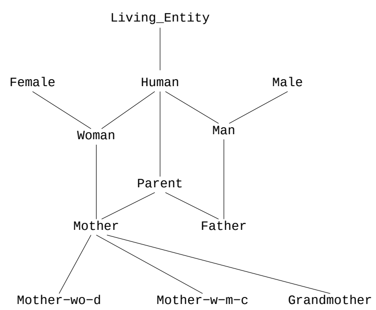
Compute all subsumption relationships (and represent them using only a minimal number of relationships)!

Useful in order to:

- check the modeling
- use the precomputed relations later when subsumption queries have to be answered

Problem can be reduced to subsumption checking: then it is a generalized sorting problem!

Example



4 General ABox Reasoning Services

- ABox Satisfiability
- Instances
- Realization and Retrieval

ABox satisfiability

Satisfiability of an ABox

Given an ABox $\mathcal{A}$, does this set of assertions have a model?

- *Notice*: ABoxes representing the real world, should always have a model.

Example

The ABox

$$X : (\forall r.\neg C), \quad Y : C, \quad (X, Y) : r$$

is not satisfiable.

ABox satisfiability in a TBox

ABox satisfiability in a TBox

Given an ABox $\mathcal{A}$ and a TBox $\mathcal{T}$, is $\mathcal{A}$ consistent with the terminology introduced in $\mathcal{T}$, i.e., is $\mathcal{T} \cup \mathcal{A}$ satisfiable?

Example

If we extend our example with

```
MARGRET: Woman
(DIANA,MARGRET): has-child,
```

then the ABox becomes unsatisfiable in the given TBox.

- Problem is reducible to satisfiability of an ABox:
  ... normalize terminology, then unfold all concept and role descriptions in the ABox

Instance relations

Instance relations

Which additional ABox formulae of the form $a\colon C$ follow logically from a given ABox and TBox?

- Is $a^{\mathcal{I}} \in C^{\mathcal{I}}$ true in all models $\mathcal{I}$ of $\mathcal{T} \cup \mathcal{A}$?
- Does the formula $C(a)$ logically follow from the translation of $\mathcal{A}$ and $\mathcal{T}$ to predicate logic?

Reductions:

- Instance relations wrt. an ABox and a TBox can be reduced to instance relations wrt. ABox: use normalization and unfolding
- Instance relations in an ABox can be reduced to ABox unsatisfiability:

$$a\colon C \text{ holds in } \mathcal{A} \iff \mathcal{A} \cup \{a\colon \neg C\} \text{ is unsatisfiable}$$

Examples

Example

- ELIZABETH: Mother-with-many-children?
  yes
- WILLIAM: ¬ Female?
  yes
- ELIZABETH: Mother-without-daughter?
  no (no CWA!)
- ELIZABETH: Grandmother?
  no (only male, but not necessarily human!)

Realization

Realization

For a given object *a*, determine the most specialized concept symbols such that *a* is an instance of these concepts

Motivation:

- Similar to classification
- Is the minimal representation of the instance relations (in the set of concept symbols)
- Will give us faster answers for instance queries!

Reduction: Can be reduced to (a sequence of) instance relation tests.

Retrieval

Retrieval

Given a concept description *C*, determine the set of all (specified) instances of the concept description.

Example

We ask for all instances of the concept Male.
For our TBOX/ABox we will get the answer CHARLES, ANDREW, EDWARD, WILLIAM.

- Reduction: Compute the set of instances by testing the instance relation for each object!
- Implementation: Realization can be used to speed this up

5 Summary and Outlook

Reasoning services – summary

- Satisfiability of concept descriptions
  - in a given TBox or in an empty TBox
- Subsumption between concept descriptions
  - in a given TBox or in an empty TBox
- Classification
- Satisfiability of an ABox
  - in a given TBox or in an empty TBox
- Instance relations in an ABox
  - in a given TBox or in an empty TBox
- Realization
- Retrieval

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Outlook

- How to determine subsumption between two concept descriptions (in the empty TBox)?
- How to determine instance relations/ABox satisfiability?
- How to implement the mentioned reductions efficiently?
- Does normalization and unfolding introduce another source of computational complexity?

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