Principles of Knowledge Representation and Reasoning
Semantic Networks and Description Logics III: Description Logics – Reasoning Services and Reductions

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Example TBox & ABox

Male $\equiv \neg$Female

Human $\sqsubseteq$ Living_entity

Woman $\equiv$ Human $\sqcap$ Female

Man $\equiv$ Human $\sqcap$ Male

Mother $\equiv$ Woman $\sqcap \exists$ has-child.Human

Father $\equiv$ Man $\sqcap \exists$ has-child.Human

Parent $\equiv$ Father $\sqcup$ Mother

Grandmother $\equiv$ Woman $\sqcap \exists$ has-child.Parent

Mother-without-daughter $\equiv$ Mother $\sqcap \forall$ has-child.Male

Mother-with-many-children $\equiv$ Mother $\sqcap (\geq 3$ has-child)

DIANA: Woman
ELIZABETH: Woman
CHARLES: Man
EDWARD: Man
ANDREW: Man

DIANA: Mother-without-daughter
(ELIZABETH, CHARLES): has-child
(ELIZABETH, EDWARD): has-child
(ELIZABETH, ANDREW): has-child
(DIANA, WILLIAM): has-child
(CHARLES, WILLIAM): has-child
Motivation: Reasoning services

What do we want to know?

- We want to check whether the knowledge base is reasonable:
  - Is each defined concept in a TBox satisfiable?
  - Is a given TBox satisfiable?
  - Is a given ABox satisfiable?

- What can we conclude from the represented knowledge?
  - Is concept $X$ subsumed by concept $Y$?
  - Is an object a instance of a concept $X$?

- These problems can be reduced to logical satisfiability or implication – using the logical semantics.

- However, we take a different route: we will try to simplify these problems and then we specify direct inference methods.
2 Basic Reasoning Services

- Satisfiability without a TBox
- Satisfiability in TBox
- Eliminating the TBox
- Normalization
- Unfolding
Satisfiability of concept descriptions

Given a concept description $C$ in “isolation”, i.e., in an empty TBox, is $C$ satisfiable?

Test:
- Does there exist an interpretation $\mathcal{I}$ such that $C^\mathcal{I} \neq \emptyset$?
- Translated into FOL: Is the formula $\exists x \, C(x)$ satisfiable?

Example

$\text{Woman} \sqcap (\leq 0 \text{ has-child}) \sqcap (\geq 1 \text{ has-child})$ is unsatisfiable.
Satisfiability of concept descriptions in a TBox

Satisfiability of concept descriptions in a TBox

Given a TBox $\mathcal{T}$ and a concept description $C$, is $C$ satisfiable?

Test:

- Does there exist a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^\mathcal{I} \neq \emptyset$?
- Translated into FOL: Is the formula $\exists x \ C(x)$ together with the formulae resulting from the translation of $\mathcal{T}$ satisfiable?

Example

Mother-without-daughter $\sqcap \forall$has-child.Female is unsatisfiable, given our previously specified family TBox.
Reduction: Getting rid of the TBox

We can reduce satisfiability problem of concept descriptions in a TBox to the satisfiability problem of concept descriptions in the empty TBox.

Idea:

- Since TBoxes are cycle-free, one can understand a concept definition as a kind of “macro”.
- For a given TBox \( \mathcal{T} \) and a given concept description \( C \), all defined concept symbols appearing in \( C \) can be expanded until \( C \) contains only undefined concept symbols.
- An expanded concept description is then satisfiable if and only if \( C \) is satisfiable in \( \mathcal{T} \).
- **Problem**: What do we do with partial definitions (using \( \sqsubseteq \))?
Normalized terminologies

- A terminology is called **normalized** when it does not contain definitions of the form $A \sqsubseteq C$.
- In order to **normalize** a terminology, replace

  $$A \sqsubseteq C$$

  by

  $$A \equiv A^* \sqcap C,$$

  where $A^*$ is a **fresh** concept symbol (not appearing elsewhere in $\mathcal{T}$).
- If $\mathcal{T}$ is a terminology, the normalized terminology is denoted by $\tilde{\mathcal{T}}$. 
Normalizing is reasonable

Theorem (Normalization invariance)

If \( \mathcal{I} \) is a model of the terminology \( \mathcal{T} \), then there exists a model \( \mathcal{I}' \) of \( \tilde{\mathcal{T}} \) such that for all concept symbols \( A \) occurring in \( \mathcal{T} \), it holds \( A^\mathcal{I} = A^{\mathcal{I}'} \), and vice versa.

Proof.

“\( \Rightarrow \)”: Let \( \mathcal{I} \) be a model of \( \mathcal{T} \). This model should be extended to \( \mathcal{I}' \) so that the freshly introduced concept symbols also get interpretations. Assume \( (A \sqsubseteq C) \in \mathcal{T} \), i.e., we have \( (A = A^* \sqcap C) \in \tilde{\mathcal{T}} \).

Then set \( A^{*\mathcal{I}'} := A^\mathcal{I} \).

\( \mathcal{I}' \) obviously satisfies \( \tilde{\mathcal{T}} \) and has the same interpretation for all symbols in \( \mathcal{T} \).

“\( \Leftarrow \)”: Given a model \( \mathcal{I}' \) of \( \tilde{\mathcal{T}} \), its restriction to symbols of \( \mathcal{T} \) is the interpretation we look for.

\( \square \)
TBox unfolding

- We say that a **normalized TBox** is **unfolded by one step** when all defined concept symbols on the right sides are replaced by their defining terms.

- **Example:** Mother $\equiv$ Woman $\sqcap \ldots$ is unfolded to Mother $\equiv$ (Human $\sqcap$ Female) $\sqcap \ldots$

- We write $U(T)$ to denote a one-step unfolding and $U^n(T)$ to denote an $n$-step unfolding.

- We say that $T$ is **unfolded** if $U(T) = T$.

- $U^n(T)$ is called the **unfolding** of $T$ if $U^n(T) = U^{n+1}(T)$. If such an unfolding exists, it is denoted by $\hat{T}$.
Properties of unfoldings (1): Existence

Theorem (Existence of unfolded terminology)

Each normalized terminology $\mathcal{T}$ can be unfolded, i.e., its unfolding $\hat{\mathcal{T}}$ exists.

Proof idea.

The main reason is that terminologies have to be cycle-free. The proof can be done by induction of the definition depth of concepts.
Properties of unfoldings (2): Equivalence

Theorem (Model equivalence for unfolded terminologies)

$I$ is a model of a normalized terminology $\mathcal{T}$ if and only if it is a model of $\hat{\mathcal{T}}$.

Proof sketch.

$\Rightarrow$ : Let $I$ be a model of $\mathcal{T}$. Then it is also a model of $U(\mathcal{T})$, since on the right side of the definitions only terms with identical interpretations are substituted. However, then it must also be a model of $\hat{\mathcal{T}}$.

$\Leftarrow$ : Let $I$ be a model for $U(\mathcal{T})$. Clearly, this is also a model of $\mathcal{T}$ (with the same argument as above). This means that any model $\hat{\mathcal{T}}$ is also a model of $\mathcal{T}$. □
Generating models

- All concept and role names not occurring on the left hand side of definitions in a terminology $\mathcal{T}$ are called primitive components.
- Interpretations restricted to primitive components are called initial interpretations.

Theorem (Model extension)

For each initial interpretation $\mathcal{I}$ of a normalized TBox, there exists a unique interpretation $\mathcal{I}$ extending $\mathcal{J}$ and satisfying $\mathcal{T}$.

Proof idea.

Use $\hat{\mathcal{T}}$ and compute an interpretation for all defined symbols.

Corollary (Model existence for TBoxes)

Each TBox has at least one model.
Unfolding of concept descriptions

- Similar to the unfolding of TBoxes, we can define the unfolding of a concept description.
- We write $\hat{C}$ for the unfolded version of $C$.

**Theorem (Satisfiability of unfolded concepts)**

An concept description $C$ is satisfiable in a terminology $\mathcal{T}$ if and only if $\hat{C}$ is satisfiable in an empty terminology.

**Proof.**

“$\Rightarrow$”: trivial.

“$\Leftarrow$”: Use the interpretation for all the symbols in $\hat{C}$ to generate an initial interpretation of $\mathcal{T}$.
Then extend it to a full model $\mathcal{I}$ of $\mathcal{T}$.
This satisfies $\mathcal{T}$ as well as $\hat{C}$. Since $\hat{C}^\mathcal{I} = C^\mathcal{I}$, it satisfies also $C$. $\square$
3 General TBox Reasoning Services

- Subsumption
- Subsumption vs. Satisfiability
- Classification
Subsumption in a TBox

Given a terminology $\mathcal{T}$ and two concept descriptions $C$ and $D$, is $C$ subsumed by (or a sub-concept of) $D$ in $\mathcal{T}$ (symb. $C \sqsubseteq_\mathcal{T} D$)?

Test:

- Is $C$ interpreted as a subset of $D$ in each model $\mathcal{I}$ of $\mathcal{T}$, i.e. $C^\mathcal{I} \subseteq D^\mathcal{I}$?
- Is the formula $\forall x \left( C(x) \rightarrow D(x) \right)$ a logical consequence of the translation of $\mathcal{T}$ into FOL?

Example

Given our family TBox, it holds Grandmother $\sqsubseteq_\mathcal{T}$ Mother.
Subsumption (without a TBox)

Given two concept descriptions $C$ and $D$, is $C$ subsumed by $D$ regardless of a TBox (or in an empty TBox) (symb. $C \sqsubseteq D$)?

Test:

- Is $C$ interpreted as a subset of $D$ for all interpretations $\mathcal{I}$ ($C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$)?
- Is the formula $\forall x (C(x) \rightarrow D(x))$ logically valid?

Example

Clearly, Human $\sqcap$ Female $\sqsubseteq$ Human.
Reductions

- Subsumption in a TBox can be reduced to subsumption in the empty TBox:
  
  ... normalize and unfold TBox and concept descriptions.

- Subsumption in the empty TBox can be reduced to unsatisfiability:
  
  ... $C \sqsubseteq D$ iff $C \cap \neg D$ is unsatisfiable.

- Unsatisfiability can be reduced to subsumption:
  
  ... $C$ is unsatisfiable iff $C \sqsubseteq (C \cap \neg C)$. 
Classification

Compute all subsumption relationships (and represent them using only a minimal number of relationships)!

Useful in order to:
- check the modeling
- use the precomputed relations later when subsumption queries have to be answered

Problem can be reduced to subsumption checking: then it is a generalized sorting problem!

Example
4 General ABox Reasoning Services

- ABox Satisfiability
- Instances
- Realization and Retrieval
ABox satisfiability

Satisfiability of an ABox

Given an ABox $\mathcal{A}$, does this set of assertions have a model?

- Notice: ABoxes representing the real world, should always have a model.

Example

The ABox

$$X : (\forall r. \neg C), \quad Y : C, \quad (X, Y) : r$$

is not satisfiable.
ABox satisfiability in a TBox

Given an ABox $\mathcal{A}$ and a TBox $\mathcal{T}$, is $\mathcal{A}$ consistent with the terminology introduced in $\mathcal{T}$, i.e., is $\mathcal{T} \cup \mathcal{A}$ satisfiable?

Example

If we extend our example with

MARGRET: Woman
(DIANA,MARGRET): has-child,

then the ABox becomes unsatisfiable in the given TBox.

- Problem is reducible to satisfiability of an ABox:
  ... normalize terminology, then unfold all concept and role descriptions in the ABox
Instance relations

Which additional ABox formulae of the form \( a : C \) follow logically from a given ABox and TBox?

- Is \( a^T \in C^T \) true in all models \( \mathcal{I} \) of \( \mathcal{T} \cup \mathcal{A} \)?
- Does the formula \( C(a) \) logically follow from the translation of \( \mathcal{A} \) and \( \mathcal{T} \) to predicate logic?

Reductions:

- Instance relations wrt. an ABox and a TBox can be reduced to instance relations wrt. ABox: use normalization and unfolding
- Instance relations in an ABox can be reduced to ABox unsatisfiability:

\[
\text{a : C holds in } \mathcal{A} \iff \mathcal{A} \cup \{ a : \neg C \} \text{ is unsatisfiable}
\]
Examples

Example

- ELIZABETH: Mother-with-many-children?
  yes

- WILLIAM: ¬ Female?
  yes

- ELIZABETH: Mother-without-daughter?
  no (no CWA!)

- ELIZABETH: Grandmother?
  no (only male, but not necessarily human!)
Realization

For a given object $a$, determine the most specialized concept symbols such that $a$ is an instance of these concepts.

Motivation:

- Similar to classification
- Is the minimal representation of the instance relations (in the set of concept symbols)
- Will give us faster answers for instance queries!

Reduction: Can be reduced to (a sequence of) instance relation tests.
Retrieval

Given a concept description \( C \), determine the set of all (specified) instances of the concept description.

Example

We ask for all instances of the concept \textit{Male}.
For our TBOX/ABox we will get the answer \textit{CHARLES}, \textit{ANDREW}, \textit{EDWARD}, \textit{WILLIAM}.

- **Reduction**: Compute the set of instances by testing the instance relation for each object!
- **Implementation**: Realization can be used to speed this up
5 Summary and Outlook
Reasoning services – summary

- Satisfiability of concept descriptions
  - in a given TBox or in an empty TBox
- Subsumption between concept descriptions
  - in a given TBox or in an empty TBox
- Classification
- Satisfiability of an ABox
  - in a given TBox or in an empty TBox
- Instance relations in an ABox
  - in a given TBox or in an empty TBox
- Realization
- Retrieval
Outlook

- How to determine subsumption between two concept descriptions (in the empty TBox)?
- How to determine instance relations/ABox satisfiability?
- How to implement the mentioned reductions efficiently?
- Does normalization and unfolding introduce another source of computational complexity?