Principles of Knowledge Representation and Reasoning

Modal Logics

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Motivation

Motivation for studying modal logics

- Notions like believing and knowing require a more general semantics than e.g. propositional logic has.
- Some KR formalisms can be understood as (fragments of) a propositional modal logic.
- Application 1: Spatial representation formalism RCC8
- Application 2: Description logics
- Application 3: Reasoning about time
- Application 4: Reasoning about actions, strategies, etc.

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⇝ How to formalize this?

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Propositional logic + operators $\Box$ & $\Diamond$ (Box & Diamond):

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- Necessarily φ (alethic)
- Always φ (temporal)
- φ should be true (deontic)
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$\rightsquigarrow$ Different semantics for different intended readings

Semantics

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- Note: There are only 4 different unary Boolean functions $\{T,F\} \to \{T,F\}$.

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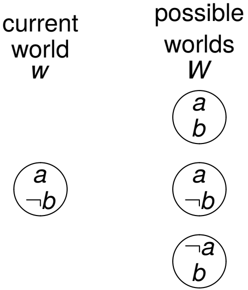
- Meanings of $\Box$ and $\Diamond$ are interrelated by: $\Diamond\varphi \equiv \neg\Box\neg\varphi$.

Semantics: an example



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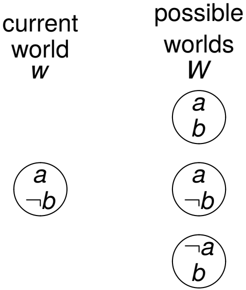

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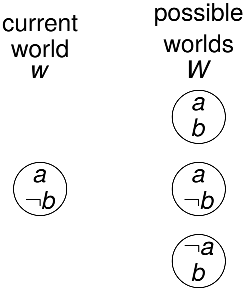

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- $\Box a$

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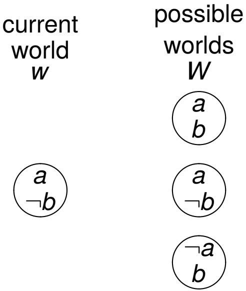

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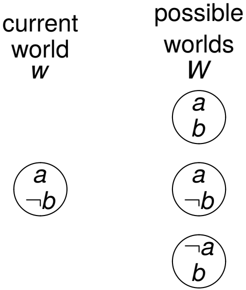

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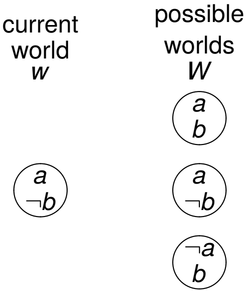

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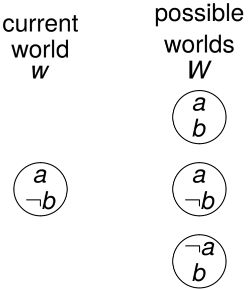

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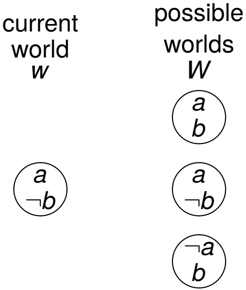

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$\rightsquigarrow$ Frames

Frames, interpretations, and worlds

Definition (Kripke frame)

A (Kripke, relational) frame is a pair $\mathcal{F} = \langle W, R \rangle$, where W is a non-empty set (of worlds) and $R \subseteq W \times W$ is a binary relation on W (accessibility relation).

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For $(w, v) \in R$ we write also $w\,R\,v$. We say that v is an R-successor of w or that v is R-reachable from w.

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Definition (Kripke model)

For a given set of propositional variables Σ, a Kripke model (or interpretation) based on the frame $\mathcal{F} = \langle W, R \rangle$ is a triple $\mathcal{I} = \langle W, R, \pi \rangle$, where π is a function that maps worlds w to truth assignments $\pi_w : \Sigma \to \{T, F\}$, i.e.:

$$\pi : W \to \{T, F\}^{\Sigma}, \ w \mapsto \pi_w.$$

Semantics: truth in a world

A formula φ is true in world w in an interpretation $\mathcal{I} = \langle W, R, \pi \rangle$ under the following conditions:

$\mathcal{I}, w \models a$	iff $\pi_w(a) = T$
$\mathcal{I}, w \models \top$	
$\mathcal{I}, w \not\models \bot$	
$\mathcal{I}, w \models \neg\varphi$	iff $\mathcal{I}, w \not\models \varphi$
$\mathcal{I}, w \models \varphi \wedge \psi$	iff $\mathcal{I}, w \models \varphi$ and $\mathcal{I}, w \models \psi$
$\mathcal{I}, w \models \varphi \vee \psi$	iff $\mathcal{I}, w \models \varphi$ or $\mathcal{I}, w \models \psi$
$\mathcal{I}, w \models \varphi \rightarrow \psi$	iff $\mathcal{I}, w \not\models \varphi$ or $\mathcal{I}, w \models \psi$
$\mathcal{I}, w \models \varphi \leftrightarrow \psi$	iff $\mathcal{I}, w \models \varphi$ if and only if $\mathcal{I}, w \models \psi$
$\mathcal{I}, w \models \Box\varphi$	iff $\mathcal{I}, u \models \varphi$, for all u s.t. wRu
$\mathcal{I}, w \models \Diamond\varphi$	iff $\mathcal{I}, u \models \varphi$, for some u s.t. wRu

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A formula φ is satisfiable in a frame $\mathcal{F}$ (satisfiable in a class of frames $\mathcal{C}$) if it is satisfiable in an interpretation $\mathcal{I}$ based on $\mathcal{F}$ (satisfiable in an interpretation $\mathcal{I}$ based on some frame contained in $\mathcal{C}$).

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A formula φ is true in an interpretation $\mathcal{I}$ (symbolically $\mathcal{I} \models \varphi$) if φ is true in all worlds of $\mathcal{I}$.

A formula φ is valid in a frame $\mathcal{F}$ or $\mathcal{F}$-valid (symb. $\mathcal{F} \models \varphi$) if φ is true in all interpretations based on $\mathcal{F}$.

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A formula φ is valid in a class of frames $\mathcal{C}$ or $\mathcal{C}$-valid (symb. $\mathcal{C} \models \varphi$) if $\mathcal{F} \models \varphi$ for all $\mathcal{F} \in \mathcal{C}$.

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4. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ (axiom schema *K*)

Moreover, it holds:

If φ is **K**-valid, then $\Box\varphi$ is **K**-valid

Validity: some examples

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If $\Box\varphi$ is true in w, then both $\Box(\varphi \to \psi)$ and $\Box\varphi$ are true in w. Hence both $\varphi \to \psi$ and φ are true in every world u accessible from w. Hence ψ is true in any such u, and therefore $w \models \Box\psi$.

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Since $\mathcal{I}$ and w were chosen arbitrarily, the argument goes through for any $\mathcal{I}, w$, i.e., K is **K**-valid. □

Non-validity: example

Proposition

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Proof.

A counterexample is the following interpretation $\mathcal{I} = \langle W, R, \pi \rangle$ with:

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We have $\mathcal{I}, w \not\models \Diamond\top$ because there is no u such that wRu. □

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We have $\mathcal{I}, w \models \Box a$, but $\mathcal{I}, w \not\models a$. □

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A counterexample is the following interpretation:

$$\mathcal{I} = \langle \{u,v,w\}, \{(u,v),(v,w)\}, \pi \rangle$$

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Hence, $\mathcal{I}, u \models \Box a$, but $\mathcal{I}, u \not\models \Box\Box a$. $\Box$

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Accessibility and axiom schemata

Let us consider the following axiom schemata:

T:	$\Box\varphi \rightarrow \varphi$	(knowledge axiom)
4:	$\Box\varphi \rightarrow \Box\Box\varphi$	(positive introspection)
5:	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$	(or $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$: negative introspection)
B:	$\varphi \rightarrow \Box\Diamond\varphi$	
D:	$\Box\varphi \rightarrow \Diamond\varphi$	(or $\Box\varphi \rightarrow \neg\Box\neg\varphi$: disbelief in the negation)

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B:	$\varphi \rightarrow \Box\Diamond\varphi$	
D:	$\Box\varphi \rightarrow \Diamond\varphi$	(or $\Box\varphi \rightarrow \neg\Box\neg\varphi$: disbelief in the negation)

… and the following classes of frames, for which the accessibility relation is restricted as follows:

- T: reflexive (wRw for each world w)
- 4: transitive (wRu and uRv implies wRv)
- 5: euclidian (wRu and wRv implies uRv)
- B: symmetric (wRu implies uRw)
- D: serial (for each w there exists v with wRv)

Correspondence between accessibility relations and axiom schemata (1)

Theorem

Axiom schema T $(4, 5, B, D)$ *is* **T***- valid (***4-, 5-, B-***, or* **D***-valid, respectively).*

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We conclude: T is true in all worlds in all interpretations based on **T**-frames. □

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Because $\mathcal{F}$ is not a **T**-frame, there is a world w such that not wRw.
Construct an interpretation $\mathcal{I}$ such that $\mathcal{I}, w \not\models a$ and $\mathcal{I}, v \models a$ for all v such that wRv.
Now $\mathcal{I}, w \models \Box a$ and $\mathcal{I}, w \not\models a$, and hence $\mathcal{I}, w \not\models \Box a \rightarrow a$. □

Different Logics

Different modal logics

Name	Property	Axiom schema
K	–	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
T	reflexivity	$\Box\varphi \rightarrow \varphi$
4	transitivity	$\Box\varphi \rightarrow \Box\Box\varphi$
5	euclidicity	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$
B	symmetry	$\varphi \rightarrow \Box\Diamond\varphi$
D	seriality	$\Box\varphi \rightarrow \Diamond\varphi$

Some basic modal logics:

$$\begin{array}{rcl} K & & \\ KT4 & = & S4 \\ KT5 & = & S5 \\ \vdots & & \end{array}$$

Different modal logics

logics	$\Box$	$\Diamond = \neg\Box\neg$	K	T	4	5	B	D
alethic	necessarily	possibly	Y	Y	Y	Y	Y	Y
epistemic	known	possible	Y	Y	Y	Y	Y	Y
doxastic	believed	possible	Y	N	Y	Y	N	Y
deontic	obligatory	permitted	Y	N	Y?	Y?	N	Y
temporal	always (in the future)	sometimes (...)	Y	Y/N	Y	N	N	N/Y

Analytic Tableaux

Proof methods

- How can we show that a formula is $\mathcal{C}$-valid?

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$\rightsquigarrow$ Method of (analytic/semantic) tableaux

Tableaux method

A tableau is a tree with nodes marked as follows:

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A tableau is constructed by using the tableau rules.

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Tableau rules for propositional logic

$$\frac{w \models \varphi \vee \psi}{w \models \varphi \mid w \models \psi} \qquad \frac{w \not\models \varphi \vee \psi}{\begin{array}{c} w \not\models \varphi \\ w \not\models \psi \end{array}} \qquad \frac{w \models \neg\varphi}{w \not\models \varphi}$$

$$\frac{w \models \varphi \wedge \psi}{\begin{array}{c} w \models \varphi \\ w \models \psi \end{array}} \qquad \frac{w \not\models \varphi \wedge \psi}{w \not\models \varphi \mid w \not\models \psi} \qquad \frac{w \not\models \neg\varphi}{w \models \varphi}$$

$$\frac{w \models \varphi \rightarrow \psi}{w \not\models \varphi \mid w \models \psi} \qquad \frac{w \not\models \varphi \rightarrow \psi}{\begin{array}{c} w \models \varphi \\ w \not\models \psi \end{array}}$$

Additional tableau rules for modal logic **K**

$$\frac{w \models \Box\varphi}{v \models \varphi}$$ if wRv is on the branch already

$$\frac{w \not\models \Box\varphi}{\begin{array}{c} wRv \\ v \not\models \varphi \end{array}}$$ for new v

$$\frac{w \models \Diamond\varphi}{\begin{array}{c} wRv \\ v \models \varphi \end{array}}$$ for new v

$$\frac{w \not\models \Diamond\varphi}{v \not\models \varphi}$$ if wRv is on the branch already

Properties of **K** tableaux

Proposition

If a K-tableau is closed, the truth condition at the root cannot be satisfied.

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*If a K-tableau with root $w \not\models \varphi$ is closed, then φ is **K**-valid.*

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If a K-tableau is closed, the truth condition at the root cannot be satisfied.

Theorem (Soundness)

*If a K-tableau with root $w \not\models \varphi$ is closed, then φ is **K**-valid.*

Theorem (Completeness)

*If φ is **K**-valid, then there is a closed tableau with root $w \not\models \varphi$.*

Termination: There are strategies for constructing **K**-tableaux that always terminate after a finite number of steps, and result in a closed tableau whenever one exists.

Tableau rules for other modal logics

Proofs within more restricted classes of frames allow the use of further tableau rules.

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- For reflexive (**T**) frames we may extend any branch with *wRw*.
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  - If *wRv* and *vRu* are in a branch, *wRu* may be added to the branch.

Tableau rules for other modal logics

Proofs within more restricted classes of frames allow the use of further tableau rules.

- For reflexive (**T**) frames we may extend any branch with wRw.
- For transitive (**4**) frames we have the following additional rule:
  - If wRv and vRu are in a branch, wRu may be added to the branch.
- For serial (**D**) frames we have the following rule:
  - If there is $w \models \ldots$ or $w \not\models \ldots$ on a branch, then add wRv for a new world v.
- Similar rules for other properties...

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Complextity of simple modal logics

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For example, one can show that each formula φ that is satisfiable in some S5-frame is satisfiable in an S5-frame with $|W| \leq |\varphi|$.

Proposition

Checking whether a modal formula is satisfiable in some S5-model is NP-complete (and hence checking S5-validity is coNP-complete).

For other modal logics, such as K, KT, KD, K4, S4, these problems are PSPACE-complete.

Embedding in FOL

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Connection between propositional modal logic and FOL?

- There are similarities between predicate logic and propositional modal logics:
  1. $\Box$ vs. $\forall$
  2. $\Diamond$ vs. $\exists$
  3. possible worlds vs. objects of the universe
- In fact, many propositional modal logics can be embedded in the predicate logic.

$\Rightarrow$ Modal logics can be understood as a sublanguage of FOL.

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Embedding modal logics into FOL (1)

1. $\tau(p, x) = p(x)$ for propositional variables p

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6. $\tau(\Diamond\varphi, x) = \exists y(R(x, y) \wedge \tau(\varphi, y))$ for some new y

Embedding modal logics into FOL (2)

Theorem

φ is K-valid if and only if $\forall x\, \tau(\varphi, x)$ is valid in FOL.

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Example

$\Box p \wedge \Diamond(p \rightarrow q) \rightarrow \Diamond q$ is K-valid, because

$$\forall x(\forall x'(R(x,x') \rightarrow p(x')) \wedge \exists x'(R(x,x') \wedge (p(x') \rightarrow q(x'))) \\ \rightarrow \exists x'(R(x,x') \wedge q(x')))$$

is valid in FOL.

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Outlook & literature

Outlook

We only looked at some basic propositional modal logics. There are also:

- modal first order logics (with quantification $\forall$ and $\exists$ and predicates)

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- modal first order logics (with quantification $\forall$ and $\exists$ and predicates)
- multi-modal logics: more than one modality, e.g. knowledge/belief operators for several agents
- temporal and dynamic logics (modalities that refer to time or programs, respectively)

Outlook

Did we really do something new? Couldn't we have done everything in propositional modal logic in FOL already?

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Outlook

Did we really do something new? Couldn't we have done everything in propositional modal logic in FOL already?

- Yes – but now we know much more about the (restricted) system and have decidable problems!

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