Principles of Knowledge Representation and Reasoning

Complexity Theory

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Motivation

Why complexity theory?

- Complexity theory can answer questions on how easy or hard a problem is
- Gives hints on what algorithms could be appropriate, e.g.:
  - algorithms for polynomial-time problems are usually easy to design
  - for NP-complete problems, backtracking and local search work well
- Gives hints on what type of algorithm will (most probably) not work
- Gives hint on what sub-problems might be interesting

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Basic Notions: a Reminder

Algorithms and Turing machines

- We use Turing machines as formal models of algorithms
- This is justified, because:
  - we assume that Turing machines can compute all computable functions
  - the resource requirements (in term of time and memory) of a Turing machine are only polynomially worse than other models
- The regular type of Turing machine is the deterministic one: DTM (or simply TM)
- Often, however, we use the notion of nondeterministic TMs: NDTM

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Problems, solutions, and complexity

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  I: instance; A: answer
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- Complexity of an algorithm: function

$$T : \mathbb{N} \to \mathbb{N},$$

  measuring the number of basic steps (or memory requirement) the algorithm needs to compute an answer depending on the size of the instance

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- Complexity of a problem: complexity of the most efficient algorithm that solves this problem.

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Complexity classes P and NP

Problems are categorized into complexity classes according to the requirements of computational resources:

- The class of problems decidable on deterministic Turing machines in polynomial time: P
  - Problems in P are assumed to be efficiently solvable (although this might not be true if the exponent is very large)
  - In practice, a reasonable definition
- The class of problems decidable on non-deterministic Turing machines in polynomial time, i.e., having a poly. length accepting computation for all positive instances: NP
- More classes are definable using other resource bounds on time and memory

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  - the technical tool here is the polynomial reduction (or any other appropriate reduction)
  - show that some hard problem can be reduced to the problem at hand

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Polynomial reduction

- Given languages L_1 and L_2, L_1 can be polynomially reduced to L_2, written $L_1 \leq_p L_2$, if there exists a polynomial time-computable function f such that

$$x \in L_1 \iff f(x) \in L_2.$$

Rationale: it cannot be harder to decide L_1 than L_2

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- L is complete for C (C-complete) if L is C-hard and $L \in C$.

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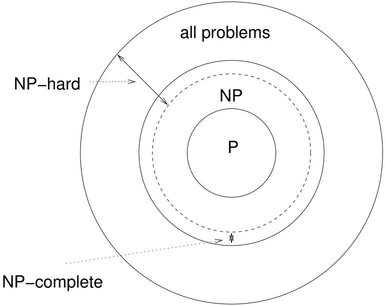

Beyond NP

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- Examples: UNSAT, TAUT $\in$ co-NP!
- *Note:* P is closed under complement, in particular,

$$\text{P} \subseteq \text{NP} \cap \text{co-NP}$$

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There are problems even more difficult than NP and co-NP…

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Definition ((N)PSPACE)

PSPACE (NPSPACE) is the class of decision problems that can be decided on deterministic (non-deterministic) Turing machines using only polynomially many tape cells.

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Some facts about PSPACE:

- PSPACE is closed under complements (... as all other deterministic classes)
- PSPACE is identical to NPSPACE (because non-deterministic Turing machines can be simulated on deterministic TMs using only quadratic space: Savitch's Theorem)
- NP $\subseteq$ PSPACE (because in polynomial time one can "visit" only polynomial space, i.e., NP $\subseteq$ NPSPACE)

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PSPACE-completeness

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An example for a PSPACE-complete problem is the NDFA equivalence problem:

Instance: Two non-deterministic finite state automata A_1 and A_2.

Question: Are the languages accepted by A_1 and A_2 identical?

Other complexity classes ...

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- There are (infinitely many) classes inside P (circuit classes with different depths)
- . . . and for most of the classes we do not know whether the containment relationships are strict

Oracle TMs and the Polynomial Hierarchy

Oracle Turing machines

- An Oracle Turing machine ((N)OTM) is a Turing machine (DTM, NDTM) with the possibility to query an oracle (i. e., a different Turing machine without resource restrictions) whether it accepts or rejects a given string.

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- Formalization:
  - a tape onto which strings for the oracle are written,
  - a yes/no answer from the oracle depending on whether it accepts or rejects the input string.
- Usage of OTMs answers what-if questions: What if we could solve the oracle-problem efficiently?

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- Turing reducibility can also be applied to general search problems!

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Complexity classes based on Oracle TMs

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Example

Consider the Minimum Equivalent Expression (MEE) problem:

Instance: A well-formed Boolean formula φ using the standard connectives (not $\leftrightarrow$) and a non-negative integer k.

Question: Is there a well-formed Boolean formula φ' that contains k or fewer literal occurrences and that is logically equivalent to φ?

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- MEE $\in$ $\mathrm{NP}^{\mathrm{NP}}$.

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The polynomial hierarchy

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The polynomial hierarchy PH

$$\begin{array}{lllllllll} \Sigma_0^p & = & \mathrm{P} & \Pi_0^p & = & \mathrm{P} & \Delta_0^p & = & \mathrm{P} \\ \Sigma_{i+1}^p & = & \mathrm{NP}^{\Sigma_i^p} & \Pi_{i+1}^p & = & \text{co-}\Sigma_{i+1}^p & \Delta_{i+1}^p & = & \mathrm{P}^{\Sigma_i^p} \end{array}$$

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$$\Sigma_0^p = P \qquad \Pi_0^p = P \qquad \Delta_0^p = P$$

$$\Sigma_{i+1}^p = NP^{\Sigma_i^p} \qquad \Pi_{i+1}^p = \text{co-}\Sigma_{i+1}^p \qquad \Delta_{i+1}^p = P^{\Sigma_i^p}$$

- PH $= \bigcup_{i\geq 0}(\Sigma_i^p \cup \Pi_i^p \cup \Delta_i^p) \subseteq$ PSPACE
- NP $= \Sigma_1^p$
- co-NP $= \Pi_1^p$

Quantified Boolean formulae: definition

- If φ is a propositional formula, P is the set of Boolean variables used in φ and σ is a sequence of $\exists p$ and $\forall p$, one for every $p \in P$, then $\sigma\varphi$ is a QBF.

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- This definition directly leads to an AND/OR tree traversal algorithm for evaluating QBF.

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Example

The formulae $\exists x \forall y (x \leftrightarrow y)$ and $\forall x \forall y (x \vee y)$ are false.

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The Polynomial Hierarchy: connection to QBF

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Special cases corresponding to SAT and TAUT:

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The Polynomial Hierarchy: connection to QBF

Truth of QBFs with prefix $\overbrace{\forall\exists\forall\dots}^{i}$ is Π_i^p-complete.

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Special cases corresponding to SAT and TAUT:

- The truth of QBFs with prefix $\exists x_1^1 \dots x_n^1$ is NP= Σ_1^p-complete.
- The truth of QBFs with prefix $\forall x_1^1 \dots x_n^1$ is co-NP= Π_1^p-complete.

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