Principles of Knowledge Representation and Reasoning

Predicate logic

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Motivation

Why first-order logic (FOL)?

- In propositional logic, the only building blocks are atomic propositions.
- We cannot talk about the internal structures of these propositions.
- Example:
  - All CS students know formal logic
  - Peter is a CS student
  - Therefore, Peter knows formal logic

  ...not possible in propositional logic
- Idea: We introduce predicates, functions, object variables and quantifiers.

Syntax

Syntax

- variable symbols: $x, y, z, \ldots$
- n-ary function symbols: $f, g, \ldots$
- constant symbols: $a, b, c, \ldots$
- n-ary predicate symbols: $P, Q, \ldots$
- logical symbols: $\forall$, $\exists$, $=$, $\neg$, $\wedge$, $\ldots$

Syntax

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- n-ary predicate symbols: $P, Q, \ldots$
- logical symbols: $\forall$, $\exists$, $=$, $\neg$, $\wedge$, …

Terms	t	::=	x	variable
		|	$f(t_1, \ldots, t_n)$	function application
		|	a	constant
Formulae	φ	::=	$P(t_1, \ldots, t_n)$	atomic formulae
		|	$t = t'$	identity formulae
		|	$\ldots$	propositional connectives
		|	$\forall x \varphi'$	universal quantification
		|	$\exists x \varphi'$	existential quantification

Ground term, etc.: term, etc. without variable occurrences

Semantics

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Semantics: idea

- In FOL, the universe of discourse consists of objects: we consider functions and relations over these objects.
- Function symbols are mapped to functions, predicate symbols are mapped to relations, and terms to objects.
- Notation: Instead of $\mathcal{I}(x)$ we write $x^{\mathcal{I}}$.
- Note: Usually one considers all possible non-empty universes. (However, sometimes the interpretations are restricted to particular domains, e.g. integers or real numbers.)
- Satisfiability and validity is then considered wrt. all these universes.

Formal semantics: interpretations

Interpretations: $\mathcal{I} = \langle \mathcal{D}, \cdot^{\mathcal{I}} \rangle$ with $\mathcal{D}$ being an arbitrary non-empty set and $\cdot^{\mathcal{I}}$ being a function which maps

- n-ary function symbols f to n-ary functions $f^{\mathcal{I}} \in [\mathcal{D}^n \to \mathcal{D}]$,
- constant symbols a to objects $a^{\mathcal{I}} \in \mathcal{D}$, and
- n-ary predicates P to n-ary relations $P^{\mathcal{I}} \subseteq \mathcal{D}^n$.

Interpretation of ground terms:

$$(f(t_1, \ldots, t_n))^{\mathcal{I}} = f^{\mathcal{I}}(t_1{}^{\mathcal{I}}, \ldots, t_n{}^{\mathcal{I}}) \ (\in \mathcal{D})$$

Truth of ground atoms:

$$\mathcal{I} \models P(t_1, \ldots, t_n) \quad \text{iff} \quad \langle t_1{}^{\mathcal{I}}, \ldots, t_n{}^{\mathcal{I}} \rangle \in P^{\mathcal{I}}$$

Examples

$$
\begin{aligned}
\mathcal{D} &= \{d_1, \ldots, d_n\}, n \geq 2 \\
\mathrm{a}^{\mathcal{I}} &= d_1 \\
\mathrm{b}^{\mathcal{I}} &= d_2 \\
\mathrm{Cat}^{\mathcal{I}} &= \{d_1\} \\
\mathrm{Red}^{\mathcal{I}} &= \mathcal{D} \\
\mathcal{I} &\models \mathrm{Red}(\mathrm{b}) \\
\mathcal{I} &\not\models \mathrm{Cat}(\mathrm{b})
\end{aligned}
$$

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$$

$$
\begin{aligned}
\mathcal{D} &= \{1, 2, 3, \dots\} \\
1^{\mathcal{I}} &= 1 \\
2^{\mathcal{I}} &= 2 \\
&\vdots \\
\mathrm{even}^{\mathcal{I}} &= \{2, 4, 6, \dots\} \\
\mathrm{succ}^{\mathcal{I}} &= \{(1 \mapsto 2), (2 \mapsto 3), \dots\} \\
\mathcal{I} &\not\models \mathrm{even}(3) \\
\mathcal{I} &\models \mathrm{even}(\mathrm{succ}(3))
\end{aligned}
$$

Formal semantics: variable assignments

V is the set of variables. Functions $\alpha\colon V \rightarrow \mathcal{D}$ are called variable assignments.

Notation: $\alpha[x/d]$ is identical to α except for x where $\alpha[x/d](x) = d$.

Interpretation of terms under $\mathcal{I}, \alpha$:

$$\begin{aligned} x^{\mathcal{I},\alpha} &= \alpha(x) \\ a^{\mathcal{I},\alpha} &= a^{\mathcal{I}} \\ (f(t_1,\ldots,t_n))^{\mathcal{I},\alpha} &= f^{\mathcal{I}}({t_1}^{\mathcal{I},\alpha},\ldots,{t_n}^{\mathcal{I},\alpha}) \end{aligned}$$

Truth of atomic formulae:

$$\mathcal{I},\alpha \models P(t_1,\ldots,t_n) \quad \text{iff} \quad \langle {t_1}^{\mathcal{I},\alpha},\ldots,{t_n}^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}}$$

Example (cont'd):
$\alpha = \{x \mapsto d_1, y \mapsto d_2\}$ $\qquad$ $\mathcal{I},\alpha \models \text{Red}(x)$ $\qquad$ $\mathcal{I},\alpha[y/d_1] \models \text{Cat}(y)$

Formal semantics: truth

Truth of φ under $\mathcal{I}$ and α ($\mathcal{I}, \alpha \models \varphi$) is defined as follows.

$$
\begin{array}{ll}
\mathcal{I}, \alpha \models P(t_1, \ldots, t_n) & \text{iff } \langle t_1^{\mathcal{I},\alpha}, \ldots, t_n^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}} \\
\mathcal{I}, \alpha \models t_1 = t_2 & \text{iff } t_1^{\mathcal{I},\alpha} = t_2^{\mathcal{I},\alpha} \\
\mathcal{I}, \alpha \models \neg\varphi & \text{iff } \mathcal{I}, \alpha \not\models \varphi \\
\mathcal{I}, \alpha \models \varphi \wedge \psi & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \varphi \vee \psi & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \varphi \rightarrow \psi & \text{iff } \text{if} \mathcal{I}, \alpha \models \varphi, \text{then} \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \varphi \leftrightarrow \psi & \text{iff } \mathcal{I}, \alpha \models \varphi \, \text{iff} \, \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \forall x \, \varphi & \text{iff } \mathcal{I}, \alpha[x/d] \models \varphi \text{ for all } d \in \mathcal{D} \\
\mathcal{I}, \alpha \models \exists x \, \varphi & \text{iff } \mathcal{I}, \alpha[x/d] \models \varphi \text{ for some } d \in \mathcal{D}
\end{array}
$$

Examples

Questions:

$$\mathcal{I}, \alpha \models \text{Cat}(b) \lor \neg\text{Cat}(b)?$$

$$\begin{aligned}
\mathcal{D} &= \{d_1, \ldots, d_n\},\ n > 1 \\
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$\mathcal{I}, \alpha \models \mathrm{Cat}(x) \rightarrow \mathrm{Cat}(x) \lor \mathrm{Cat}(y)$?

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No

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No

$\mathcal{I}, \alpha \models \text{Cat}(a) \land \text{Cat}(b)$?

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$\mathcal{I},\alpha \models \text{Cat}(x) \rightarrow \text{Cat}(y)$?
No

$\mathcal{I},\alpha \models \text{Cat}(a) \wedge \text{Cat}(b)$?
Yes

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Questions:

$\mathcal{I}, \alpha \models \text{Cat}(b) \lor \neg\text{Cat}(b)$?
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$\mathcal{I}, \alpha \models \text{Cat}(x) \rightarrow \text{Cat}(x) \lor \text{Cat}(y)$?
Yes

$\mathcal{I}, \alpha \models \text{Cat}(x) \rightarrow \text{Cat}(y)$?
No

$\mathcal{I}, \alpha \models \text{Cat}(a) \land \text{Cat}(b)$?
Yes

$\mathcal{I}, \alpha \models \forall x(\text{Cat}(x) \rightarrow \text{Red}(x))$?

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$$\Theta = \left\{ \begin{array}{l} \mathrm{Cat}(a), \mathrm{Cat}(b) \\ \forall x(\mathrm{Cat}(x) \rightarrow \mathrm{Red}(x)) \end{array} \right\}$$

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$\mathcal{I}, \alpha \models \mathrm{Cat}(x) \rightarrow \mathrm{Cat}(y)$?
No

$\mathcal{I}, \alpha \models \mathrm{Cat}(a) \land \mathrm{Cat}(b)$?
Yes

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Yes

$\mathcal{I}, \alpha \models \Theta$?

Examples

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$\mathcal{I}, \alpha \models \mathrm{Cat}(a) \wedge \mathrm{Cat}(b)$?
Yes

$\mathcal{I}, \alpha \models \forall x(\mathrm{Cat}(x) \rightarrow \mathrm{Red}(x))$?
Yes

$\mathcal{I}, \alpha \models \Theta$? Yes

Terminology

$\mathcal{I}, \alpha$ is a model of φ iff

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A formula can be satisfiable, unsatisfiable, falsifiable, valid, ...

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Formulae φ and ψ are logically equivalent (symb.: $\varphi \equiv \psi$) iff for all $\mathcal{I}, \alpha$:

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Note: $\mathrm{P(x)} \not\equiv \mathrm{P(y)}$!

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Logical implication is also analogous to propositional logic:

$$\Theta \models \varphi \quad \text{iff} \quad \text{for all } \mathcal{I}, \alpha \text{ s.t. } \mathcal{I}, \alpha \models \Theta \text{ also } \mathcal{I}, \alpha \models \varphi.$$

Free and bound variables

Variables can be free or bound (by a quantifier) in a formula:

$$
\begin{aligned}
\mathrm{free}(x) &= \{x\} \\
\mathrm{free}(f(t_1,\ldots,t_n)) &= \mathrm{free}(t_1) \cup \cdots \cup \mathrm{free}(t_n) \\
\mathrm{free}(t_1 = t_2) &= \mathrm{free}(t_1) \cup \mathrm{free}(t_2) \\
\mathrm{free}(P(t_1,\ldots,t_n)) &= \mathrm{free}(t_1) \cup \cdots \cup \mathrm{free}(t_n) \\
\mathrm{free}(\neg\varphi) &= \mathrm{free}(\varphi) \\
\mathrm{free}(\varphi * \psi) &= \mathrm{free}(\varphi) \cup \mathrm{free}(\psi), \text{ for } * = \vee, \wedge, \rightarrow, \leftrightarrow \\
\mathrm{free}(Qx\varphi) &= \mathrm{free}(\varphi) \setminus \{x\}, \text{ for } Q = \forall, \exists
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Example: $\forall x (R(y,z) \wedge \exists y (\neg P(y,x) \vee R(y,z)))$

Which occurrences are free, which are not free?

Open & closed formulae

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- Note that logical equivalence, satisfiability, and entailment are independent from variable assignments if we consider only closed formulae.
- For closed formulae, we omit α in connection with $\models$:

$$\mathcal{I} \models \varphi.$$

Normal forms

Prenex Normal Form

The prenex normal form of a FOL formula has the following form:

quantifier prefix + (quantifier free) matrix

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Generate prenex normal form:

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Theorem

For each FOL formula, an equivalent formula in prenex normal form exists and can be effectively computed.

Skolemization

We can further simplify formulae by eliminating existential quantifiers using fresh function symbols (Skolem functions).

Theorem (Skolem normal form)

Let φ be a closed formula in prenex normal form with all variables pairwise distinct of the form $\varphi = \forall x_1 \ldots \forall x_i \exists y \, \psi$. Let g_i be an i-ary function symbols not appearing in φ.

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Then φ is satisfiable iff

$$\varphi' = \forall x_1 \dots \forall x_i \, \psi[y/g_i(x_1, \dots, x_i)]$$

is satisfiable.

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Proof idea.

For each assignment to $x_1 \ldots x_i$, there is a value of $y[= g(x_1, \ldots, x_i)]$ and vice versa.

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Prenex normal form without existential quantifiers.

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Example

$$\exists x\, ((\forall x\, p(x)) \land \neg q(x))$$

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$$\exists x\, ((\forall x\, p(x)) \land \neg q(x))$$
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$$\exists y\,((\forall x\, p(x)) \land \neg q(y))$$
$$\exists y\,(\forall x\,(p(x) \land \neg q(y)))$$
$$\forall x\,(p(x) \land \neg q(g_0))$$

Herbrand interpretations

Reducing FOL satisfiability to propositional satisfiability ...

Idea 1: We use one particular interpretation which has as the universe of discourse all possible ground terms – and we add one constant if we do not have already one $\rightsquigarrow$ Herbrand universe

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$$a^{\mathcal{I}} = a$$

$$(f(t_1,\ldots,t_n))^{\mathcal{I}} = f(t_1,\ldots,t_n)$$

$\mathcal{I}$ could then be defined such that, e.g., $\mathcal{I} \not\models P(a_0,a_0)$, $\mathcal{I} \not\models P(a_0, g_2(a_0,a_0))$, etc.

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Theorem

A formula φ is satisfiable if $E(\varphi)$ is satisfiable.

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- Semi-decision method for unsatisfiability
- In fact, unsatisfiability (and validity) in FOL is only semi-decidable (use e.g. PCP to prove)!

Further Theorems

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Some corollaries from the previous theorems:

Theorem (Compactness)

Let $\Phi \cup \{\psi\}$ be a set of closed formulae.

(a) $\Phi \models \psi$ *iff there exists a finite subset* $\Phi' \subseteq \Phi$ *s. t.* $\Phi' \models \psi$.

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Theorem (Löwenheim-Skolem)

Each countable set of closed formulae that is satisfiable is satisfiable on a countable domain.

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