Principles of Knowledge Representation and Reasoning

Predicate logic

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April 24, 2018

1 Motivation

Why first-order logic (FOL)?

- In propositional logic, the only building blocks are atomic propositions.
- We cannot talk about the internal structures of these propositions.
- Example:
  - All CS students know formal logic
  - Peter is a CS student
  - Therefore, Peter knows formal logic

  …not possible in propositional logic
- Idea: We introduce predicates, functions, object variables and quantifiers.

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2 Syntax

Syntax

- variable symbols: $x, y, z, \ldots$
- n-ary function symbols: $f, g, \ldots$
- constant symbols: $a, b, c, \ldots$
- n-ary predicate symbols: $P, Q, \ldots$
- logical symbols: $\forall$, $\exists$, $=$, $\neg$, $\wedge$, $\ldots$

Terms	t	$::= x$	variable
		$\mid\ f(t_1, \ldots, t_n)$	function application
		$\mid\ a$	constant
Formulae	φ	$::= P(t_1, \ldots, t_n)$	atomic formulae
		$\mid\ t = t'$	identity formulae
		$\mid\ \ldots$	propositional connectives
		$\mid\ \forall x \varphi'$	universal quantification
		$\mid\ \exists x \varphi'$	existential quantification

Ground term, etc.: term, etc. without variable occurrences

3 Semantics

- Interpretations
- Variable Assignments
- Definition of Truth
- Terminology
- Free and Bound Variables
- Open and Closed Formulae

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Semantics: idea

- In FOL, the universe of discourse consists of objects: we consider functions and relations over these objects.
- Function symbols are mapped to functions, predicate symbols are mapped to relations, and terms to objects.
- Notation: Instead of $\mathcal{I}(x)$ we write $x^{\mathcal{I}}$.
- Note: Usually one considers all possible non-empty universes. (However, sometimes the interpretations are restricted to particular domains, e.g. integers or real numbers.)
- Satisfiability and validity is then considered wrt. all these universes.

Formal semantics: interpretations

Interpretations: $\mathcal{I} = \langle \mathcal{D}, \cdot^{\mathcal{I}} \rangle$ with $\mathcal{D}$ being an arbitrary non-empty set and $\cdot^{\mathcal{I}}$ being a function which maps

- n-ary function symbols f to n-ary functions $f^{\mathcal{I}} \in [\mathcal{D}^n \to \mathcal{D}]$,
- constant symbols a to objects $a^{\mathcal{I}} \in \mathcal{D}$, and
- n-ary predicates P to n-ary relations $P^{\mathcal{I}} \subseteq \mathcal{D}^n$.

Interpretation of ground terms:

$$(f(t_1, \ldots, t_n))^{\mathcal{I}} \quad = \quad f^{\mathcal{I}}({t_1}^{\mathcal{I}}, \ldots, {t_n}^{\mathcal{I}}) \ \ (\in \mathcal{D})$$

Truth of ground atoms:

$$\mathcal{I} \models P(t_1, \ldots, t_n) \quad \text{iff} \quad \langle {t_1}^{\mathcal{I}}, \ldots, {t_n}^{\mathcal{I}} \rangle \in P^{\mathcal{I}}$$

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Examples

$$
\begin{aligned}
\mathcal{D} &= \{d_1, \ldots, d_n\}, n \geq 2 \\
\mathrm{a}^{\mathcal{I}} &= d_1 \\
\mathrm{b}^{\mathcal{I}} &= d_2 \\
\mathrm{Cat}^{\mathcal{I}} &= \{d_1\} \\
\mathrm{Red}^{\mathcal{I}} &= \mathcal{D} \\
\mathcal{I} &\models \mathrm{Red}(\mathrm{b}) \\
\mathcal{I} &\not\models \mathrm{Cat}(\mathrm{b})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D} &= \{1, 2, 3, \ldots\} \\
1^{\mathcal{I}} &= 1 \\
2^{\mathcal{I}} &= 2 \\
&\vdots \\
\mathrm{even}^{\mathcal{I}} &= \{2, 4, 6, \ldots\} \\
\mathrm{succ}^{\mathcal{I}} &= \{(1 \mapsto 2), (2 \mapsto 3), \ldots\} \\
\mathcal{I} &\not\models \mathrm{even}(3) \\
\mathcal{I} &\models \mathrm{even}(\mathrm{succ}(3))
\end{aligned}
$$

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Formal semantics: variable assignments

V is the set of variables. Functions $\alpha\colon V \to \mathcal{D}$ are called variable assignments.

Notation: $\alpha[x/d]$ is identical to α except for x where $\alpha[x/d](x) = d$.

Interpretation of terms under $\mathcal{I}, \alpha$:

$$\begin{array}{rcl} x^{\mathcal{I},\alpha} & = & \alpha(x) \\ a^{\mathcal{I},\alpha} & = & a^{\mathcal{I}} \\ (f(t_1,\ldots,t_n))^{\mathcal{I},\alpha} & = & f^{\mathcal{I}}({t_1}^{\mathcal{I},\alpha},\ldots,{t_n}^{\mathcal{I},\alpha}) \end{array}$$

Truth of atomic formulae:

$$\mathcal{I},\alpha \models P(t_1,\ldots,t_n) \quad \text{iff} \quad \langle {t_1}^{\mathcal{I},\alpha},\ldots,{t_n}^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}}$$

Example (cont'd):
$\alpha = \{x \mapsto d_1, y \mapsto d_2\}$ $\qquad$ $\mathcal{I},\alpha \models \mathrm{Red}(x)$ $\qquad$ $\mathcal{I},\alpha[y/d_1] \models \mathrm{Cat}(y)$

Formal semantics: truth

Truth of φ under $\mathcal{I}$ and α ($\mathcal{I}, \alpha \models \varphi$) is defined as follows.

$$
\begin{array}{ll}
\mathcal{I}, \alpha \models P(t_1, \ldots, t_n) & \text{iff } \langle t_1^{\mathcal{I},\alpha}, \ldots, t_n^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}} \\
\mathcal{I}, \alpha \models t_1 = t_2 & \text{iff } t_1^{\mathcal{I},\alpha} = t_2^{\mathcal{I},\alpha} \\
\mathcal{I}, \alpha \models \neg\varphi & \text{iff } \mathcal{I}, \alpha \not\models \varphi \\
\mathcal{I}, \alpha \models \varphi \wedge \psi & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \varphi \vee \psi & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \varphi \rightarrow \psi & \text{iff if} \mathcal{I}, \alpha \models \varphi, \text{then} \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \varphi \leftrightarrow \psi & \text{iff } \mathcal{I}, \alpha \models \varphi \,\text{iff}\, \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \forall x\, \varphi & \text{iff } \mathcal{I}, \alpha[x/d] \models \varphi \text{ for all } d \in \mathcal{D} \\
\mathcal{I}, \alpha \models \exists x\, \varphi & \text{iff } \mathcal{I}, \alpha[x/d] \models \varphi \text{ for some } d \in \mathcal{D}
\end{array}
$$

Examples

$$
\begin{aligned}
\mathcal{D} &= \{d_1, \ldots, d_n\},\ n > 1 \\
a^{\mathcal{I}} &= d_1 \\
b^{\mathcal{I}} &= d_1 \\
\mathrm{Cat}^{\mathcal{I}} &= \{d_1\} \\
\mathrm{Red}^{\mathcal{I}} &= \mathcal{D} \\
\alpha &= \{(x \mapsto d_1), (y \mapsto d_2)\}
\end{aligned}
$$

$$
\Theta = \left\{ \begin{array}{l} \mathrm{Cat}(a), \mathrm{Cat}(b) \\ \forall x(\mathrm{Cat}(x) \rightarrow \mathrm{Red}(x)) \end{array} \right\}
$$

Questions:

$\mathcal{I}, \alpha \models \mathrm{Cat}(b) \vee \neg \mathrm{Cat}(b)$?
Yes

$\mathcal{I}, \alpha \models \mathrm{Cat}(x) \rightarrow \mathrm{Cat}(x) \vee \mathrm{Cat}(y)$?
Yes

$\mathcal{I}, \alpha \models \mathrm{Cat}(x) \rightarrow \mathrm{Cat}(y)$?
No

$\mathcal{I}, \alpha \models \mathrm{Cat}(a) \wedge \mathrm{Cat}(b)$?
Yes

$\mathcal{I}, \alpha \models \forall x(\mathrm{Cat}(x) \rightarrow \mathrm{Red}(x))$?
Yes

$\mathcal{I}, \alpha \models \Theta$? Yes

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Terminology

$\mathcal{I}, \alpha$ is a model of φ iff

$$\mathcal{I}, \alpha \models \varphi.$$

A formula can be satisfiable, unsatisfiable, falsifiable, valid, ...

Formulae φ and ψ are logically equivalent (symb.: $\varphi \equiv \psi$) iff for all $\mathcal{I}, \alpha$:

$$\mathcal{I}, \alpha \models \varphi \text{ iff } \mathcal{I}, \alpha \models \psi.$$

Note: $\mathrm{P(x)} \not\equiv \mathrm{P(y)}$!

Logical implication is also analogous to propositional logic:

$$\Theta \models \varphi \quad \text{iff} \quad \text{for all } \mathcal{I}, \alpha \text{ s.t. } \mathcal{I}, \alpha \models \Theta \text{ also } \mathcal{I}, \alpha \models \varphi.$$

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Free and bound variables

Variables can be free or bound (by a quantifier) in a formula:

$$
\begin{aligned}
\mathrm{free}(x) &= \{x\} \\
\mathrm{free}(f(t_1,\dots,t_n)) &= \mathrm{free}(t_1) \cup \dots \cup \mathrm{free}(t_n) \\
\mathrm{free}(t_1 = t_2) &= \mathrm{free}(t_1) \cup \mathrm{free}(t_2) \\
\mathrm{free}(P(t_1,\dots,t_n)) &= \mathrm{free}(t_1) \cup \dots \cup \mathrm{free}(t_n) \\
\mathrm{free}(\neg\varphi) &= \mathrm{free}(\varphi) \\
\mathrm{free}(\varphi * \psi) &= \mathrm{free}(\varphi) \cup \mathrm{free}(\psi),\ \text{for } * = \vee, \wedge, \rightarrow, \leftrightarrow \\
\mathrm{free}(Qx\varphi) &= \mathrm{free}(\varphi) \setminus \{x\},\ \text{for } Q = \forall, \exists
\end{aligned}
$$

Example: $\forall x(R(y,z) \wedge \exists y(\neg P(y,x) \vee R(y,z)))$

Which occurrences are free, which are not free?

Open & closed formulae

- Formulae without free variables are called closed formulae or sentences. Formulae with free variables are called open formulae.
- Closed formulae are all we need when we want to state something about the world. Open formulae (and variable assignments) are only necessary for technical reasons (semantics of $\forall$ and $\exists$).
- Note that logical equivalence, satisfiability, and entailment are independent from variable assignments if we consider only closed formulae.
- For closed formulae, we omit α in connection with $\models$:

$$\mathcal{I} \models \varphi.$$

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4 Normal forms

Prenex Normal Form

The prenex normal form of a FOL formula has the following form:

quantifier prefix + (quantifier free) matrix

Generate prenex normal form:

1. Eliminate $\rightarrow$ and $\leftrightarrow$.
2. Move $\neg$ inside.
3. Moving quantifiers out (using a number of equivalences).

Theorem

For each FOL formula, an equivalent formula in prenex normal form exists and can be effectively computed.

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Skolemization

We can further simplify formulae by eliminating existential quantifiers using fresh function symbols (Skolem functions).

Theorem (Skolem normal form)

Let φ be a closed formula in prenex normal form with all variables pairwise distinct of the form $\varphi = \forall x_1 \ldots \forall x_i \exists y \psi$. Let g_i be an i-ary function symbols not appearing in φ.
Then φ is satisfiable iff

$$\varphi' = \forall x_1 \ldots \forall x_i \psi[y / g_i(x_1, \ldots, x_i)]$$

is satisfiable.

Proof idea.

For each assignment to $x_1 \ldots x_i$, there is a value of $y[= g(x_1, \ldots, x_i)]$ and vice versa.

Skolem normal form

Skolem Normal Form

Prenex normal form without existential quantifiers.

Notation: φ^* is SNF of φ

Theorem

For each closed formula φ, a corresponding SNF φ^* can be effectively computed.

Example

$$
\begin{array}{l}
\exists x\,((\forall x\, p(x)) \wedge \neg q(x)) \\
\exists y\,((\forall x\, p(x)) \wedge \neg q(y)) \\
\exists y\,(\forall x\,(p(x) \wedge \neg q(y))) \\
\forall x\,(p(x) \wedge \neg q(g_0))
\end{array}
$$

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5 Herbrand interpretations

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Reducing FOL satisfiability to propositional satisfiability …

Idea 1: We use one particular interpretation which has as the universe of discourse all possible ground terms – and we add one constant if we do not have already one $\leadsto$ Herbrand universe

Example: $\forall x \forall y(\neg P(x,y) \vee R(g_2(x,y),x))$
$\mathcal{D}^H = \{a_0, g_2(a_0,a_0), g_2(a_0,g_2(a_0,a_0)), \dots\}$

Idea 2: Function symbols are interpreted syntactically, predicate symbols are interpreted arbitrarily over this universe (each ground atom gets a truth value): $\leadsto$ Herbrand interpretation

$$a^{\mathcal{I}} = a$$

$$(f(t_1,\dots,t_n))^{\mathcal{I}} = f(t_1,\dots,t_n)$$

$\mathcal{I}$ could then be defined such that, e.g., $\mathcal{I} \not\models P(a_0,a_0)$, $\mathcal{I} \not\models P(a_0,g_2(a_0,a_0))$, etc.

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Herbrand models and Herbrand expansions

Theorem

A formula φ has a model iff it has a Herbrand model.

Idea 3: We expand each SNF-formula by substituting all variables by all possible terms $\rightsquigarrow$ Herbrand expansion ($E(\varphi)$)

Example: $\neg P(a_0, a_0) \vee R(g_2(a_0, a_0), a_0), \neg P(a_0, g_2(a_0, a_0)) \vee R(g_2(a_0, g_2(a_0, a_0)), a_0), \ldots$

Theorem

A formula φ is satisfiable if $E(\varphi)$ is satisfiable.

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A reduction to a satisfiability problem with infinitely many formulae

- Note that the Herbrand universe can be infinite, therefore $E(\varphi)$ can be infinite!
- If the Herbrand base is finite there is no problem (well, ...)
- Use $E(\varphi)$ in a "lazy" way, expand only as needed
- Semi-decision method for unsatisfiability
- In fact, unsatisfiability (and validity) in FOL is only semi-decidable (use e.g. PCP to prove)!

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6 Further Theorems

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Further theorems

Some corollaries from the previous theorems:

Theorem (Compactness)

Let $\Phi \cup \{\psi\}$ be a set of closed formulae.

(a) $\Phi \models \psi$ *iff there exists a finite subset* $\Phi' \subseteq \Phi$ *s. t.* $\Phi' \models \psi$.

(b) Φ *is satisfiable iff each finite subset* $\Phi' \subseteq \Phi$ *is satisfiable.*

Theorem (Löwenheim-Skolem)

Each countable set of closed formulae that is satisfiable is satisfiable on a countable domain.

7 Literature

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