Principles of Knowledge Representation and Reasoning

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Exercise Sheet 2

Due: May 3rd, 2018

Exercise 2.1 (Horn clauses, 2+2)

(a) Consider a satisfiable Horn formula ψ. Consider the interpretation in which a variable is true if and only if it is true in all models of ψ. Prove that this interpretation is also a model of ψ.

(b) Apply (a) in order to show that there exists a formula which has no logically equivalent Horn formula.

Exercise 2.2 (Predicate logic, 2+2)

(a) Classify the following expressions as terms, ground terms, atoms, formulae, sentences, or statements in meta language. If there is more than one possibility for an expression please list them all. The usage of symbols complies with the convention introduced with the syntax of predicate logic.

(a) $P(x, a)$
(b) $g(a, h(b, c))$
(c) $\mathcal{I} \not\models P(a, f(b))$
(d) $\mathcal{I}, \alpha \models P(f(x), f(a))$
(e) $g(f(y), a)$
(f) $Q(b)$ is falsifiable.
(g) $\forall x(P(x, y) \rightarrow Q(x)) \vee \neg P(y, x)$
(h) $\forall x \forall y(P(x, y) \wedge Q(x) \vee P(f(y), x))$
(i) $\forall x(\exists y(P(x, y) \wedge Q(x)) \vee P(x, y))$
(j) $Q(a) \vee P(a, b) \equiv P(b, a) \vee Q(b)$

(b) Consider the following theory:

$$\Theta = \left\{ \begin{array}{l} \forall x \neg P(x, x) \\ \forall x \forall y \forall z(P(x, y) \wedge P(y, z) \rightarrow P(x, z)) \\ \forall x \exists y P(x, y) \\ \neg \exists y P(y, a) \end{array} \right\}$$

Specify an interpretation $\mathcal{I} = \langle \mathcal{D}, \cdot^{\mathcal{I}} \rangle$ with $\mathcal{I} \models \Theta$ (with proof). Does Θ have a model that is defined on a finite domain D?

Exercise 2.3 (Formula Game and Reduction, 2+2)

(a) The Formula Game is a two-player game played on a given quantified Boolean formula (in prenex normal form) $Q_1 p_1 \ldots Q_k p_k \psi$. The rules are simple: If the outermost unassigned variable p_i is universally (existentially) quantified, it is the turn of player U (player E resp.) who assigns a truth value to that variable p_i. Thus both players finally construct a truth assignment I

to the variables occurring in the matrix formula ψ. Player E wins the game if $I(\psi) = 1$; otherwise, player U wins the game.

Check whether one of the players U or E has a strategy for winning the formula game for the following formulae:

(a) $\forall p \forall q \exists r \forall s \big((p \land r) \rightarrow (q \land s)\big)$

(b) $\forall p \exists q \exists r \big((p \rightarrow q) \land (q \rightarrow \neg r) \land (r \lor \neg p)\big)$

(b) We consider the following two-player game $\mathcal{G}$ played on a directed graph $\langle V, A\rangle$ with a designated start node $v_0 \in V$. Player 1 and player 2 choose in turn some arc in the graph such that each chosen arc starts in the head of the previously chosen arc. Player 1 begins with choosing an arc starting in node v_0. A player looses the game if s/he is unable to choose an arc to a not yet visited node in the graph.

Show that the following problem is PSPACE-complete.

Instance: A directed graph $\langle V, A\rangle$, a start node v_0.

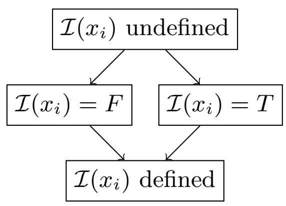
Question: Does Player 1 have a strategy for winning $\mathcal{G}$?

Hint: Existence of a winning strategy in the formula game (see exercise 3.1) is known to be PSPACE-complete even for QBF of the following form:

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \ldots \exists x_{2k-1} \forall x_{2k} \exists x_{2k+1} \psi,$$

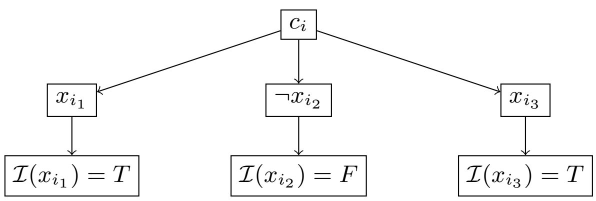
where ψ is a 3-CNF formula. For the reduction construct for a given formula of this form a directed graph. The following subgraphs will be useful:

- For each propositional variable introduce a subgraph with four nodes that represents that a variable has been assigned a truth value.



  The current player will have to decide on the truth value of the next unassigned variable x_i. Note that the node corresponding to the chosen assignment may not be revisited in the game.

- Furthermore introduce nodes for each clause c_i of ψ and the literals $l_{i_1}, \ldots, l_{i_3}$ occurring in it. For example, if $c_i = x_{i_1} \lor \neg x_{i_2} \lor x_{i_3}$:



Finally discuss the size of your graph and relate the winning strategies in the games.