Motivation

House Allocation Problem

1 Motivation

Mechanisms without Money

Motivation 1:
- According to Gibbard-Satterthwaite: In general, nontrivial social choice functions manipulable.
- One way out: Introduction of money (cf. VCG mechanisms)
- Other way out: Restriction of preferences (cf. single-peaked preferences; this chapter)

Motivation 2:
- Introduction of central concept from cooperative game theory: the core

Examples:
- House allocation problem
- Stable matchings

2 House Allocation Problem

- Definitions
- Top Trading Cycle Algorithm
House Allocation Problem

- **Players** $N = \{1, \ldots, n\}$.
- Each player $i$ owns house $i$.
- Each player $i$ has **strict linear preference order** $\prec_i$ over the set of houses.
  - **Example:** $j \prec_i k$ means player $i$ prefers house $k$ to house $j$.
- **Alternatives** $A$: allocations of houses to players (permutations $\pi \in S_n$ of $N$).
  - **Example:** $\pi(i) = j$ means player $i$ gets house $j$.
- **Objective:** reallocate the houses among the agents “appropriately”.

Note on preference relations:
- Arbitrary (strict linear) preference orders $\prec_i$ over houses, but no arbitrary preference orders $\preceq_i$ over $A$.
- **Rather:** Player $i$ **indifferent** between different allocations $\pi_1$ and $\pi_2$ as long as $\pi_1(i) = \pi_2(i)$.
  - Indifference denoted as $\pi_1 \approx_i \pi_2$.
- If player $i$ is not indifferent: $\pi_1 \prec_i \pi_2$ iff $\pi_1 \prec_i \pi_2$ for all $i \in M$ and $\pi_1 \prec_i \pi_2$ for at least one $i \in M$.

Important new aspect of house allocation problem:
- Players control resources to be allocated.
- Allocation can be subverted by subset of agents breaking away and trading among themselves.
- How to avoid such allocations?
- How to make allocation mechanism non-manipulable?

Notation: For $M \subseteq N$, let

$$A(M) = \{\pi \in A \mid \forall i \in M: \pi(i) \in M\}$$

be the set of allocations that can be achieved by the agents in $M$ trading among themselves.

Definition (blocking coalition)
Let $\pi \in A$ be an allocation. A set $M \subseteq N$ is called a **blocking coalition** for $\pi$ if there exists a $\pi' \in A(M)$ such that
- $\pi \preceq_i \pi'$ for all $i \in M$ and
- $\pi \prec_i \pi'$ for at least one $i \in M$. 

Notation: $\pi \preceq_i \pi'$ for all $i \in M$ and $\pi \prec_i \pi'$ for at least one $i \in M$. 

This makes Gibbard-Satterthwaite inapplicable.
**Intuition:**
A blocking coalition can receive houses everyone from the coalition likes at least as much as under allocation $\pi$, with at least one player being strictly better off, by trading among themselves.

**Definition (core)**
The set of allocations that is not blocked by any subset of agents is called the core.

**Question:** Is the core nonempty?

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**Top Trading Cycle Algorithm (TTCA)**

**Pseudocode:**

- let $\pi(i) = i$ for all $i \in N$.
- while players unaccounted for do
  - consider subgraph $G'$ of $G$ where each vertex has only one outgoing arc: the least-colored one from $G$.
  - identify cycles in $G'$.
  - add corresponding cyclic permutations to $\pi$.
  - delete players accounted for and incident edges from $G$.
- end while
- output $\pi$.

**Notation:**
Let $N_i$ be the set of vertices on cycles identified in iteration $i$.

**Example:**
- Player 1: 3 $\prec_i 1 \prec_i 4 \prec_i 2$
- Player 2: 4 $\prec_i 2 \prec_i 3 \prec_i 1$
- Player 3: 3 $\prec_i 4 \prec_i 2 \prec_i 1$
- Player 4: 1 $\prec_i 4 \prec_i 2 \prec_i 3$

**Corresponding graph:**

**Iteration 1:** $\pi(1) = 2$, $\pi(2) = 1$.
**Iteration 2:** $\pi(3) = 4$, $\pi(4) = 3$.
**Done:** $\pi(1) = 2$, $\pi(2) = 1$, $\pi(3) = 4$, $\pi(4) = 3$. 
**Top Trading Cycle Algorithm (TTCA)**

**Theorem**
The core of the house allocation problem consists of exactly one matching.

**Proof sketch**
**At most one matching:** Show that if a matching is in the core, it must be the one returned by the TTCA.

In TTCA, each player in $N_1$ receives his favorite house.

Therefore, $N_1$ would form a blocking coalition to any allocation that does not assign to all of those players the houses they would receive in TTCA.

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Stable Matchings

Problem statement:
- Given disjoint finite sets $M$ of men and $W$ of women.
- Assume WLOG that $|M| = |W|$ (introduce dummy-men/dummy-women).
- Each $m \in M$ has strict preference ordering $\prec_m$ over $W$.
- Each $w \in W$ has strict preference ordering $\prec_w$ over $M$.
- Matching: “appropriate” assignment of men to women such that each man is assigned to at most one woman and vice versa.

Note: A group of players can subvert a matching by opting out.

Definition (stability, blocking pair)
A matching is called unstable if there are two men $m, m'$ and two women $w, w'$ such that
- $m$ is matched to $w$,
- $m'$ is matched to $w'$, and
- $w \prec_m w'$ and $m' \prec_w m$.
The pair $\langle m, w' \rangle$ is called a blocking pair. A matching that has no blocking pairs is called stable.

Definition (core)
The core of the matching game is the set of all stable matchings.

Example:
- Man 1: $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2: $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3: $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1: $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2: $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3: $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

Two matchings:
- Matching $\{\langle m_1, w_1 \rangle, \langle m_2, w_2 \rangle, \langle m_3, w_3 \rangle\}$ unstable (\langle m_1, w_2 \rangle is a blocking pair)
- Matching $\{\langle m_1, w_1 \rangle, \langle m_3, w_2 \rangle, \langle m_2, w_3 \rangle\}$ stable

Question: Is there always a stable matching?
Answer: Yes! And it can even be efficiently constructed.

How? Deferred acceptance algorithm!

Note: A group of players can subvert a matching by opting out.
Deferred Acceptance Algorithm

Definition (deferred acceptance algorithm, male proposals)

1. Each man proposes to his top-ranked choice.
2. Each woman who has received at least one proposal (including tentatively kept one from earlier rounds) tentatively keeps top-ranked proposal and rejects rest.
3. If no man is left rejected, stop.
4. Otherwise, each man who has been rejected proposes to his top-ranked choice among the women who have not rejected him. Then, goto 2.

Deferred Acceptance Algorithm

Example:

- Man 1: \( w_3 \prec m_1 w_1 \prec m_1 w_2 \)
- Man 2: \( w_2 \prec m_2 w_3 \prec m_2 w_1 \)
- Man 3: \( w_3 \prec m_3 w_2 \prec m_3 w_1 \)
- Woman 1: \( m_2 \prec w_1 m_3 \prec w_1 m_1 \)
- Woman 2: \( m_2 \prec w_1 m_1 \prec w_2 m_3 \)
- Woman 3: \( m_2 \prec w_1 m_3 \prec w_3 m_1 \)

Deferred acceptance algorithm:

1. \( m_1 \) proposes to \( w_2, m_2 \) to \( w_1, \) and \( m_3 \) to \( w_1. \)
2. \( w_1 \) keeps \( m_3 \) and rejects \( m_2, w_2 \) keeps \( m_1. \)
3. \( m_2 \) now proposes to \( w_3. \)
4. \( w_3 \) keeps \( m_2. \)

Resulting matching: \( \{ (m_1, w_2), (m_2, w_3), (m_3, w_1) \} \).

Deferred Acceptance Algorithm

Theorem

The deferred acceptance algorithm with male proposals terminates in a stable matching.

Proof

Suppose not.

Then there exists a blocking pair \( (m_1, w_1) \) with \( m_1 \) matched to some \( w_2 \) and \( w_1 \) matched to some \( m_2. \)

Since \( (m_1, w_1) \) is blocking and \( w_2 \prec m_1 w_1, \) in the proposal algorithm, \( m_1 \) would have proposed to \( w_1 \) before \( w_2. \)

Since \( m_1 \) was not matched with \( w_1 \) by the algorithm, it must be because \( w_1 \) received a proposal from a man she ranked higher than \( m_1. \)
Deferred Acceptance Algorithm

Proof (ctd.)
Since the algorithm matches her to \( m_2 \) it follows that \( m_1 \prec_w m_2 \).
This contradicts the fact that \( \langle m_1, w_1 \rangle \) is a blocking pair.

Analogous version where the women propose: outcome would also be a stable matching.

Deferred Acceptance Algorithm

Definitions

Deferred Acceptance Algorithm

The mechanism associated with the (fe)male-proposal algorithm cannot be manipulated by the (fe)males.

4 Summary

Theorem
The stable matching produced by the (fe)male-proposal deferred acceptance algorithm is (fe)male-optimal.

In general, there is no stable matching that is male-optimal and female-optimal.
Summary

- **Avoid Gibbard-Satterthwaite** by restricting domain of preferences.

- **House allocation** problem:
  - Solved using top trading cycle algorithm.
  - Algorithm finds unique solution in the core, where no blocking coalition of players has an incentive to break away.
  - The top trading cycle mechanism cannot be manipulated.

- **Stable matchings**:
  - Solved using deferred acceptance algorithm.
  - Algorithm finds a stable matching in the core, where no blocking pair of players has an incentive to break away.
  - The mechanism associated with the (fe)male-proposal algorithm cannot be manipulated by the (fe)males.