Game Theory

9. Extensive Games with Imperfect Information

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1 Motivation

Motivation

- So far: All state information is completely known by all players
- Often in practice: Only partial knowledge (e.g. card games)
- Extensive games with imperfect information model such situations using information sets, which are sets of histories.
- Idea: Decision points are now information sets.
- Strategies: Mixed (over pure strategies) or behavioral (collections of independent mixed decisions for each information set)
- Different from incomplete information games, in which there is uncertainty about the utility functions of the other players.

2 Definitions

Definition (Extensive game)

An extensive game is a tuple $\Gamma = \langle N, H, P, f_c, (\mathscr{I}_i)_{i \in N}, (u_i)_{i \in N} \rangle$ that consists of:

- A finite non-empty set N of players.
- A set H of (finite or infinite) sequences, called histories, such that
  - it contains the empty sequence $\langle \rangle \in H$,
  - H is closed under prefixes: if $\langle a^1, \dots, a^k \rangle \in H$ for some $k \in \mathbb{N} \cup \{\infty\}$, and $l < k$, then also $\langle a^1, \dots, a^l \rangle \in H$, and
  - H is closed under limits: if for some infinite sequence $\langle a^i \rangle_{i=1}^{\infty}$, we have $\langle a^i \rangle_{i=1}^{k} \in H$ for all $k \in \mathbb{N}$, then $\langle a^i \rangle_{i=1}^{\infty} \in H$.

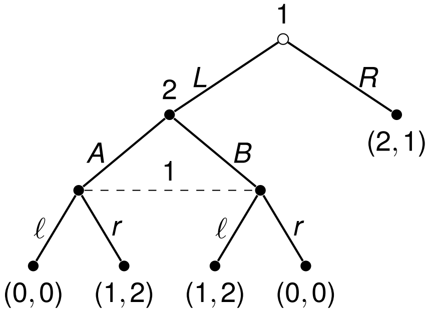
  All infinite histories and all histories $\langle a^i \rangle_{i=1}^{k} \in H$, for which there is no a^{k+1} such that $\langle a^i \rangle_{i=1}^{k+1} \in H$ are called terminal histories Z. Components of a history are called actions.

Definition (Extensive game, ctd.)

- A player function $P : H \setminus Z \to N \cup \{c\}$ that determines which player's turn it is to move after a given nonterminal history, c signifying a chance move.
- $f_c(\cdot|h)$ is a probability distribution over $A(h)$.
- $\mathscr{I}_i$ is the information partition for player i of $\{h \in H | P(h) = i\}$ with the property that $A(h) = A(h')$ whenever h and h' are in the same member of the partition. Members of the partition $I_i \in \mathscr{I}_i$ are called information sets.
- For each player $i \in N$, a utility function (or payoff function) $u_i : Z \to \mathbb{R}$ defined on the set of terminal histories.

Γ is finite, if H is finite; finite horizon, if histories are bounded.

Example



After player 1 chooses L, player 2 makes a move (A or B) player 1 cannot observe.

Simultaneous moves

- We have already chance moves, but could we extend the model with simultaneous moves as well?
- Actually, we can already model them somehow.
- In the example game after the history $\langle L \rangle$, we have essentially a simultaneous move of player 1 and 2:
  - When player 2 moves, he does not know what player 1 will do.
  - After player 2 has made his move, player 1 does not know whether A or B was chosen.
  - Only after both players have acted, they are presented with the outcome.

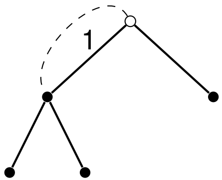
3 Recall

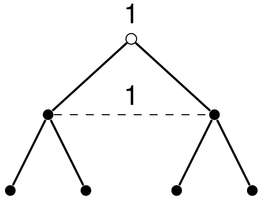
Recall

Motivation
Definitions
Recall
Strategies and outcomes
Solution Concepts
Summary

Information sets can be arbitrary. However, often we want to assume that agents always remember what they have learned before and which actions they have performed: Perfect recall.

Example (Imperfect recall)





- Left: Player 1 forgets that he made a move!
- Right: Player 1 cannot remember what his last move was.

Experience record

Definition (Experience record)

Given a history h of an extensive game, the function $X_i(h)$ is the sequence consisting of information sets that player i encounters in h and the actions that player i takes at them. X_i is called the experience record of player i in h.

Example

In our example game, Player 1 encounters two information sets in the history $h = \langle L, A \rangle$, namely $\langle \rangle$ and $\{\langle L, A \rangle, \langle L, B \rangle\}$. In the first information set, he chooses L. So $X_1(h) = \langle \langle \rangle, L, \{\langle L, A \rangle, \langle L, B \rangle\} \rangle$.

Perfect recall

Definition (Perfect Recall)

An extensive game has perfect recall if for each player i, we have $X_i(h) = X_i(h')$ whenever the histories h and h' are in the same information set of player i.

Example

In our example game, the only non-singleton information set satisfies the condition, since for $h = \langle L, A \rangle$ and $h' = \langle L, B \rangle$ we have $X_1(h) = X_1(h') = \langle \langle \rangle, L, \{\langle L, A \rangle, \langle L, B \rangle\} \rangle$. For the imperfect recall examples, the actions are different for the two histories ending up in the non-singleton information set.

In most cases, our games will have perfect recall.

4 Strategies and outcomes

Pure strategies

Definition (Pure strategy in an extensive game)

A pure strategy of a player i in an extensive game $\Gamma = \langle N, H, P, f_c, (\mathscr{I}_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a function s_i that assigns an action from $A(I_i)$ to each information set I_i.

Remark: Note that the outcome of a strategy profile s is now a probability distribution (because of the chance moves).

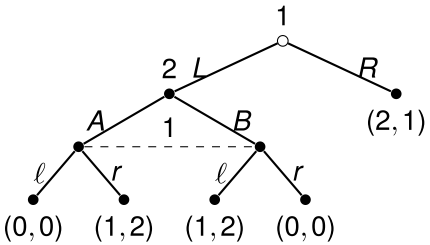
Remark: Because of the chance moves and because of the imperfect information, it probably makes more sense to consider randomized strategies.

Mixed and behavioral strategies

Definition (Mixed and behavioral strategies)

A mixed strategy of a player i in an extensive game $\Gamma = \langle N, H, P, f_c, (\mathscr{I}_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a probability distribution over the set of i's pure strategies. A behavioral strategy of player i is a collection $(\beta_i(I_i))_{I_i \in \mathscr{I}_i}$ of independent probability distributions, where $\beta_i(I_i)$ is a probability distribution over $A(I_i)$. For any history $h \in I_i \in \mathscr{I}_i$ and action $a \in A(h)$, we denote by $\beta_i(h)(a)$ the probability $\beta_i(I_i)(a)$ assigned by $\beta_i(I_i)$ to action a.

Example



- Player 1 has four pure strategies (two information sets, two actions at each).
- A mixed strategy is a probability distribution over those.
- A behavioral strategy is a pair of probability distributions, one for $\{\langle\rangle\}$ and one for $\{\langle L,A\rangle,\langle L,B\rangle\}$.

Outcomes

The outcome of a (mixed or behavioral) strategy profile σ is a probability distribution over histories $O(\sigma)$, resulting from following the individual strategies.

- For any history $h = \langle a^1, \ldots, a^k \rangle$ define a pure strategy s_i of i to be consistent with h if for any subhistory $h' = \langle a^1, \ldots, a^\ell \rangle$ with $P(h') = i$ we have $s_i(h') = a^{\ell+1}$.
- For any history, let $\pi_i(h)$ be the sum of probabilities of pure strategies s_i from σ_i consistent with h.
- Then for any mixed profile σ, the probability that $O(\sigma)$ assigns to a terminal history h is: $\prod_{i \in N \cup \{c\}} \pi_i(h)$.
- For any behavioral profile β, the probability that $O(\beta)$ assigns to $h = \langle a^1, \ldots, a^K \rangle$ is: $\prod_{k=0}^{K-1} \beta_{P(\langle a^1, \ldots, a^k \rangle)}(\langle a^1, \ldots, a^k \rangle)(a^{k+1})$.

Outcome equivalence

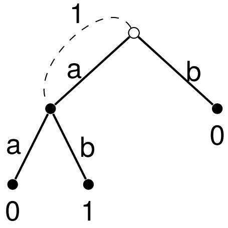
Definition

Two (mixed or behavioral) strategies of a player i are called outcome-equivalent if for every partial profile of pure strategies of the other players, the two strategies induce the same outcome.

Question: Can we find outcome-equivalent mixed strategies for behavioral strategies and vice versa?
Partial answer: Sometimes.

Counter-example (1)

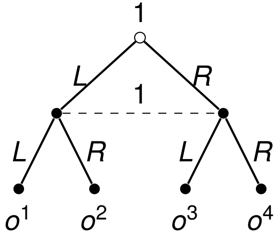
Example (Behavioral strategy without a mixed strategy)



- A behavioral strategy assigning non-zero probability to *a* and *b* generates outcomes $\langle a, a\rangle$, $\langle a, b\rangle$, and $\langle b\rangle$ with non-zero probability
- Since there are only the pure strategies playing *a* or *b*, no mixed strategy can produce $\langle a, b\rangle$.

Counter-example (2)

Example (Mixed strategy without a behavioral strategy)



- Mix the two pure strategies *LL* and *RR* equally, resulting in the distribution $(1/2, 0, 0, 1/2)$.
- No behavioral strategy can accomplish this.

Equivalence of behavioral and mixed strategies

If we restrict ourselves to games with perfect recall, however, everything works.

Theorem (Equivalence of mixed and behavioral strategies (Kuhn))

In a game of perfect recall, any mixed strategy of a given agent can be replaced by an equivalent behavioral strategy, and any behavioral strategy can be replaced by an equivalent mixed strategy.

5 Solution Concepts

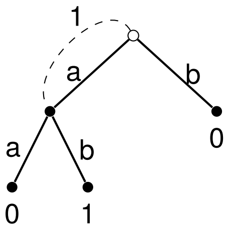
- Assessments
- Sequential Rationality
- Sequential equilibrium
- Examples

Expected utility

Similar to the case of mixed strategies for strategic games, we define the utility for mixed and behavioral strategies as expected utility, summing over all histories:

$$U_i(\sigma) = \sum_{h \in H} u_i(h) \cdot O(\sigma)(h)$$

Example



- Mixed strategy (mixing a and b) σ: $U_1(\sigma) = 0$.
- Behavioral strategy β with $p = 1/2$ for a: $U_1(\beta) = 1/4$.

Definition (Nash equilibrium in mixed strategies)

A Nash equilibrium in mixed strategies is a profile σ^* of mixed strategies with the property that for every player i:

$$U_i(\sigma^*_{-i}, \sigma^*_i) \geq U_i(\sigma^*_{-i}, \sigma_i) \text{ for every mixed strategy } \sigma_i \text{ of } i.$$

Note: Support lemma applies here as well.

Definition (Nash equilibrium in behavioral strategies)

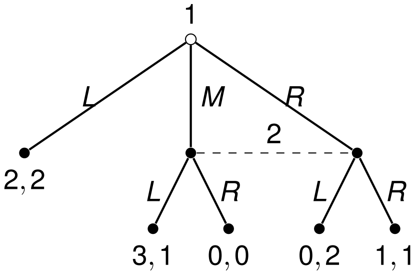
A Nash equilibrium in behavioral strategies is a profile β^* of mixed strategies with the property that for every player i:

$$U_i(\beta^*_{-i}, \beta^*_i) \geq U_i(\beta^*_{-i}, \beta_i) \text{ for every behavioral strategy } \beta_i \text{ of } i.$$

Remark: Equivalent, provided we have perfect recall.

Eliminating imperfect equilibria

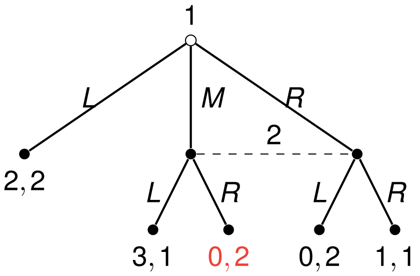
Example



Nash equilibria: (M,L) and (L,R)
Unreasonable ones: (L,R), because in the info set of player 2, *L* dominates *R*

How have we got here?

Example



Nash equilibria: (L,R)
What should player 2 do, when he ends up in his info set?
Depends on his belief: if probability that M has been played $\geq 1/2$, then R is optimal, otherwise L.

Assessments

Let us take the beliefs about what has been played into account when defining an equilibrium.

Definition (Assessment)

An assessment in an extensive game is a pair (β, μ), where β is a profile of behavioral strategies and μ is a function that assigns to every information set a probability distribution on the set of histories in the information set.

$\mu(I)(h)$ is the probability that player $P(I)$ assigns to the history $h \in I$, given I is reached. We have to modify the outcome function. Let $h^* = \langle a^1, \dots, a^K \rangle$ be a terminal history. Then:

- $O(\beta, \mu | I)(h*) = 0$, if there is no subhistory of h^+ in I,
- $O(\beta, \mu | I)(h*) =$ $\mu(I)(h) \cdot \prod_{k=L}^{K-1} \beta_{P(\langle a^1,\dots,a^k \rangle)}(\langle a^1, \dots, a^k \rangle)(a^{k+1})$, if the subhistory $\langle, a^1, \dots, a^L \rangle$ of h^* is in I with $L < K$.

Sequential rationality

Similar to O, we extend U_i: $U_i(\beta,\mu|I_i) = O(\beta,\mu|I)(h*) \cdot u_i(h*)$.

Definition (Sequential rationality)

Let Γ be an extensive game with perfect recall. The assessment (β,μ) is sequentially rational if for every player i and every information $I_i \in \mathscr{I}_i$ we have

$$U_i(\beta,\mu|I_i) \geq U_i((\beta_{-i},\beta_i'),\mu|I_i) \text{ for every } \beta_i' \text{ of } i.$$

Note: μ could be arbitrary!

Consistency with strategies

We would at least require that the beliefs are consistent with the strategies, meaning they should be derived by the strategies.

In our earlier example, player 2's belief should be derived from the behavioral strategy of player 1. E.g., the probality that M has been played should be:

$$\mu(\{\langle M\rangle, \langle R\rangle\})(M) = \beta_1(\langle\rangle)(M) / \Big(\beta_1(\langle\rangle)(M) + \beta_1(\langle\rangle)(R)\Big).$$

In other words, we use Bayes' rule to determine μ. However, what to do when the denominator is 0?

Structural consistency

By viewing an assessment as a limit of a sequence of completely mixed strategy profiles (all strategies are in the support), one can enforce the Bayes' condition also on information set that are not reached by an equlibrium profile.

Definition (Structural consistency)

Let Γ by a finite extensice game with perfect recall. An assessment (β,μ) is structural consistent if there is a sequence $((\beta^n,\mu^n))_{n=1}^{\infty}$ of assessments that converges to (β,μ) in Euclidian space and has the properties that each strategy profile β^n is completely mixed and that each belief system μ_n is derived from β_n using Bayes' rule.

Note: Kreps (1990) wrote: "a lot of bodies are buried in this definition."

Sequential equilibrium

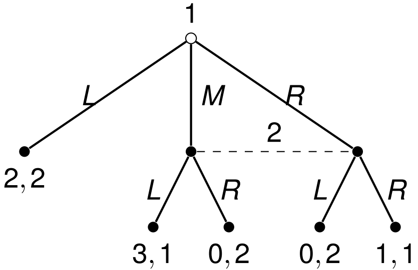
Definition (Sequential equilibrium)

An assessment is a sequential equilibrium of a finite extensice game with perfect recall if is is sequentially rational and structural consistent.

Note: There is always at least one such equlibrium.

Note: In an extensive game with perfect information, (β, μ) is sequential equlibrium iff β is a subgame-perfect equilibrium.

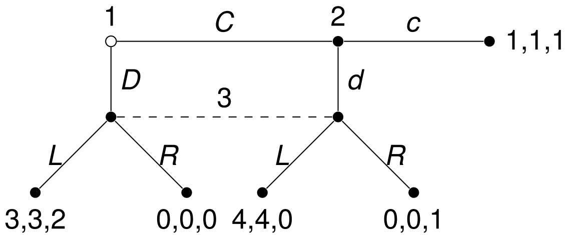
Example



Let (β,μ) be as follows: $\beta_1(L) = 1$, $\beta_2(R) = 1$, $\mu(\{\langle M\rangle,\langle R\rangle\})(M) = \alpha$ for $0 \leq \alpha \leq 1$. (β,μ) is consistent since $\beta_1^\varepsilon = (1-\varepsilon, \alpha\varepsilon, (1-\alpha)\varepsilon)$, $\beta_2^\varepsilon = (\varepsilon, 1-\varepsilon)$, and $\mu^\varepsilon(\{\langle M\rangle,\langle R\rangle\})(M) = \alpha$ converges to (β,μ) for $\varepsilon \to 0$. For $\alpha \geq 1/2$, (β,μ) is sequentially rational.

Example 2: Selten's horse

Example

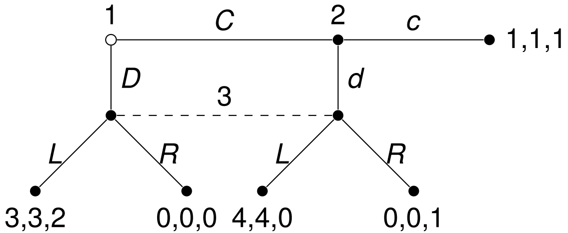


Two types of NE (for $I = \{\langle D\rangle, \langle C, d\rangle\}$):

1. $\beta_1(\langle\rangle)(D) = 1$, $1/3 \leq \beta_2(\langle C\rangle)(c) \leq 1$, $\beta_3(I)(L) = 1$
2. $\beta_1(\langle\rangle)(C) = 1$, $\beta_2(\langle C\rangle)(c) = 1$, $3/4 \leq \beta_3(I)(R) \leq 1$.

Are these also sequential equilibria?

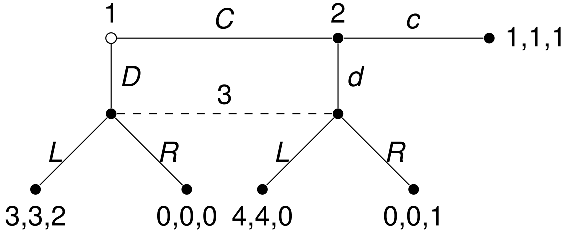
Example



$\beta_1(\langle\rangle)(D) = 1$, $1/3 \leq \beta_2(\langle C\rangle)(c) \leq 1$, $\beta_3(I)(L) = 1$ violates sequential rationality for player 2!

Selten's horse: Type 2 Nash Equilibrium

Example



For each NE, $\beta_1(\langle\rangle)(C) = 1$, $\beta_2(\langle C\rangle)(c) = 1$, $3/4 \leq \beta_3(I)(R) \leq 1$, there exists a sequential equilibrium (β, μ) with $\mu(I)(D) = 1/3$.

For consistency consider: $\beta_1^\varepsilon(\langle\rangle)(C) = 1 - \varepsilon$,
$\beta_2^\varepsilon(\langle C\rangle)(c) = 2\varepsilon/(1-\varepsilon)$, $\beta_3^\varepsilon(I)(R) = \beta_3(I)(R) - \varepsilon$.
Note: $\beta_1^\varepsilon(\langle\rangle)(D) + (\beta_1^\varepsilon(\langle\rangle)(C) \cdot \beta_2^\varepsilon(\langle C\rangle)(d)) = 3\varepsilon$.

6 Summary

Summary

- Extensive games with imperfect information can model situations, in which the player know only part of the world.
- Modeled by information sets, which are the histories, an agent cannot distinguish.
- Perfect recall requires that agents remember know what they have done and learned.
- Without it, a number of results do not hold.
- Strategies can be mixed or behavioral, which is equivalnt in the case of perfect recall.
- Nash equlibria can be defined this way, however, similiar to perfect information games, are not always reasonable.
- Sequential equilibria are the is the refinement, which is based an assessments.