Game Theory

7. Repeated Games

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May 30, 2018

1 Motivation

Motivation
Repeated Prisoners Dilemma
Strategies and Preferences in Infinite Games
Analysis of Infinite PD
Punishments and Enforceable Outcomes

Motivation and Intuition

- Remember: The *Prisoner's Dilemma* leads to the unsatisfying result because there is neither experience nor future encounters.
- What if the game is played repeatedly?
- Model this as an *extensive game* where in each turn, we repeat a given base game.
- Will *social norms* evolve?
- Will punishments, which can lead to short-term costs, nevertheless be played (are these potential punishments credible threats?)

See: `http://ncase.me/trust/`

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2 Repeated Prisoners Dilemma

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Reminder

	C	*D*
C	3,3	0,4
D	4,0	1,1

(D,D) (i.e. both players defect) is the unique Nash equilibrium, the pair of maximinimizers and the pair of strictly dominating strategies.

So, in a single encounter, there is no argument for rationally playing C!

Finitely Repeated PD

- Assume we play PD a pre-specified number of times k.
- What will be a subgame perfect equilibrium?
- Use backward induction:
  - (D,D) is the NE in the last subgame, since this would be the only NE in the one-shot game.
  - So, (D,D) will also be played in period $k-1$.
  - ...
  - So, the (only) subgame-perfect equilibrium and the only NE of this repeated game is (D,D), (D,D), ..., (D,D).

Infinitely Repeated PD

- If we play PD infinitely often, we need to solve two problems:
  1. How to define a strategy?
  2. How to define the payoff or preference?
- How to specify a strategy using only finite resources?
  - In general: One could use an *algorithm*.
  - Usually done in game theory: use Moore automata, i.e., finite state automata, where the inputs are actions of the other players, and in each state, a response action to the previous actions is generated.
  - A Nash equilibrium would then be a profile of automata (strategies) such that no deviation would be profitable.

3 Strategies and Preferences in Infinite Games

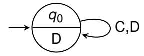
Some Possible Strategies

Using Moore automata, we can specify what to do in response to the new input (action played by others) and the state we are in. Since the automata are finite, we have only finite memory!

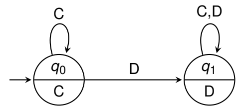
- Unconditionally *cooperative*: Always play C.
- Unconditonally *uncooperative*: Always play D.
- *Bipolar*: Start with *D* and then always exchange between C and D.
- *Tit-for-Tat*: Start with *C* and then reply with *C* to each *C* and *D* to each *D*.
- *Grim*: Start with *C*. After any play of *D* play *D* in the future forever.

Strategies as Automata

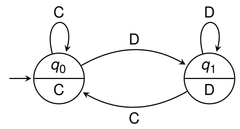
Uncooperative



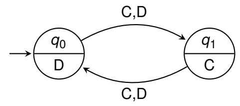
Grim



Tit-for-tat



Bipolar



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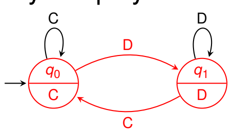
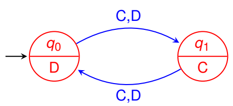
Game traces

Player 1 plays Tit-for-tat, player 2 plays bipolar; 4 rounds.





Round	Action	Utility	Accumulated payoff
1	(C,D)	(0,4)	(0,4)
2	(D,C)	(4,0)	(4,4)
3	(C,D)	(0,4)	(4,8)
4	(D,C)	(4,0)	(8,8)

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Preferences over payoffs (1)

How to define the payoff of an infinite game or whether to prefer one outcome over another one?

Given two infinite sequences $(v_i^t)_{t=1}^{\infty}$ and $(w_i^t)_{t=1}^{\infty}$, we will define when the first is prefered over the second by player i: $(v_i^t)_{t=1}^{\infty} \succsim_i (w_i^t)_{t=1}^{\infty}$

- A common method is discounting by a *discount factor* $\delta \in (0,1)$:

$$(v_i^t)_{t=1}^{\infty} \succsim_i (w_i^t)_{t=1}^{\infty} \quad \text{iff} \quad \sum_{t=1}^{\infty} \delta^{t-1}(v_i^t - w_i^t) \geq 0$$

- Overtaking is an alternative (every period counts the same):

$$(v_i^t)_{t=1}^{\infty} \succsim_i (w_i^t)_{t=1}^{\infty} \quad \text{iff} \quad \liminf_{T\to\infty} \sum_{t=1}^{T}(v_i^t - w_i^t) \geq 0$$

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Preferences over payoffs (2)

- Instead one could require that on average the payoff is better, which leads to the limit of means criterion:

$$(v_i^t)_{t=1}^{\infty} \succsim_i (w_i^t)_{t=1}^{\infty} \quad \text{iff} \quad \lim_{T\to\infty} \inf \sum_{t=1}^{T} (v_i^t - w_i^t)/T \geq 0$$

- In this case, $lim_{T\to\infty} \sum_{t=1}^{T} v_i^t / T$ would be considered the payoff, if the limit exists.

Example Preferences

- $(1,-1,0,0,\dots) \succ (0,0,0,0,\dots)$ for any $\delta \in (0,1)$ under *discounting*, but not under *limit of means* and *overtaking*.
- $(-1,2,0,0,\dots) \succ (0,0,0,0,\dots)$ under *overtaking*, but for the *limit of means* there is no preference.
- $(0,\dots,0,1,1,1,\dots) \succ (1,0,0,0,\dots)$, where in the first sequence after m zeros there are only ones, under the *limit of means*. However, for every δ there exists an m^* such that for all $m > m^*$ the preference under *discounting* is the other way around.

Infinitly Repeated Games Formally

Definition (Infinitely repeated game of G)

Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a strategic game with $A = \prod_{i\in N} A_i$. An *infinitely repeated game of G* is an extensive game with perfect information and simultaneous moves $\langle N, H, P, (\succsim_i)_{i\in N}\rangle$ in which

- $H = \{\langle\rangle\} \cup (\bigcup_{t=1}^{\infty} A^t) \cup A^{\infty}$,
- $P(h) = N$ for all nonterminal histories $h \in H$,
- $\succsim_i$ is a preference relation on A^{∞} that is based on *discounting*, *limit of means* or *overtaking*.

4 Analysis of Inifinite PD

Infinite PD Again (1)

Let us consider the Grim strategy. Is (Grim, Grim) a Nash equilibrium under all the preference criteria?

- *Overtaking* and *limit of means*: Yes! Any deviation will result in getting ≤ 1 instead of 3 infinitely often.
- *Discounting*: This is a bit more complicated.
  - If a player gets v for every round, he will accumulate the following payoff:

$$v + \delta v + \delta^2 v + \ldots = \sum_{i=1}^{\infty} \delta^{i-1} v.$$

  - Since we know that $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ (for $0 < x < 1$), we have:

$$\sum_{i=1}^{\infty} \delta^{i-1} v = v \sum_{i=0}^{\infty} \delta^i = \frac{v}{1-\delta}.$$

Infinite PD Again (2)

- Assume now, both players play *Grim* for the first k rounds. Then player 1 deviates and plays D once. In the remainder he must play D in order to get at least 1 in each round.
  - Starting in round k, he receives:

$$4 + \delta + \delta^2 + \ldots = 3 + \sum_{i=0}^{\infty} \delta^i = 3 + \frac{1}{1-\delta}$$

  - If he had not deviated, the accumulated payoff starting at round k would have been:

$$3 + 3\delta + 3\delta^2 + \ldots = 3\sum_{i=0}^{\infty} \delta^i = \frac{3}{1-\delta}.$$

  - So, a deviation is profitable if $3 + 1/(1-\delta) > 3/(1-\delta)$.
  - When $\delta < 1/3$ the deviation is profitable, i.e., for $\delta \geq 1/3$, *Grim* is a NE strategy.

Infinite PD Again (3)

- Under which preference criteria is *Tit-for-tat* an equilibrium strategy?
- *Limit of means*: Finitely many deviations do not change the payoff profile in the limit. Infinitely many deviations lead to less payoff. So Tit-for-tat is an NE strategy under this preference criterion.
- *Overtaking*: Even only one deviation leads to a payoff of 5 over two rounds instead of 6. So in no case, a deviation can lead to a better payoff.
- *Discounting*: Deviating only in one move in round k and then returning to be cooperative leads in round k to $4 + 0 + \ldots$ instead to $3 + \delta 3 + \ldots$.
  - The deviation is profitable if $4 > 3 + \delta 3$.
  - This means, we only get a profitable deviation is $\delta < 1/3$; and this is the best case for a deviation!
  - So *Tit-for-tat* is NE if $\delta \geq 1/3$.

5 Punishments and Enforceable Outcomes

Punishments

- Observe that the NE strategies are based on being able to punish a deviating player.

Definition (Minmax payoff)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ a strategic game. Player i's minimax payoff in G, also written as $v_i(G)$, is the lowest payoff that the other players can force upon player i:

$$v_i(G) = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_{-i}, a_i).$$

- The idea is that the other players all punish a deviating player in the next round(s) and allow him only to get v_i.

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Enforceable Payoffs

Definition (Feasible Payoff Profile)

Given a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a vector $v \in \mathbb{R}^N$ is called payoff profile of G if there exists $a \in A$ such that $v = u(a)$. $v \in \mathbb{R}$ is called feasible payoff profile if there exists a vector $(\alpha_a)_{a \in A} \in \mathbb{Q}^A$ with $\sum \alpha_a = 1$ and $v = \sum \alpha_a u(a)$.

- Note: Such payoffs can be generated in a repeated game by playing β_a rounds a in a set of γ games with $\gamma = \sum_{a \in A} \beta_a$ and $\alpha_a = \beta_a / \gamma$.

Definition (Enforceable payoff)

Given a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a payoff profile w with $w_i \geq v_i(G)$ for all $i \in N$ is called enforceable. If $w_i > v_i(G)$ for all $i \in N$, it is said to be strictly enforceable.

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Many different Equilibria …

- Using the concept of enforceable payoffs, one can construct many different Nash equilibria and payoff profiles for PD!
- E.g., $(3+1/3, 2)$ is a feasible payoff profile, because $4 \times (C, C)$ and $2 \times (D, C)$ leads to $((4 \times 3 + 2 \times 4)/6, (4 \times 3 + 2 \times 0)/6)$.
- Construct two automata that implement this repeated sequence and in case of deviation revert to playing D.
- These two automatons implement Nash equilibria strategies, since deviating leads to a payoff of 1 instead of $3+1/3$ or 2!
- Folk theorems stating that all enforceable outcomes are reachable have been proven for the general case.

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Summary

- Repeated games are *extensive games* with simultaneous moves, in which a base strategic game is played in each round.
- Strategies are described using finite Moore automata.
- For preferences over the payoffs of infinitely repeated games, different preference criteria are possible.
- In the repeated *Prisoners Dilemma*, it is possible to play Nash Equilibrium strategies that result in (C,C) sequences.
- In fact, it is possible to achieve any possible feasible payoff profile in the limit of means criterion.

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