Game Theory

3. Mixed Strategies

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Mixed Strategies

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Definitions
Support Lemma
Nash's Theorem
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Mixed Strategies

Observation: Not every strategic game has a pure-strategy Nash equilibrium (e. g. matching pennies).

Question:

- Can we do anything about that?
- Which strategy to play then?

Idea: Consider randomized strategies.

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Notation

Let X be a set.

Then $\Delta(X)$ denotes the set of probability distributions over X.

That is, each $p \in \Delta(X)$ is a mapping $p : X \to [0, 1]$ with

$$\sum_{x \in X} p(x) = 1.$$

Mixed Strategies

A mixed strategy is a strategy where a player is allowed to randomize his action (throw a dice mentally and then act according to what he has decided to do for each outcome).

Definition (Mixed strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A mixed strategy of player i in G is a probability distribution $\alpha_i \in \Delta(A_i)$ over player i's actions.

For $a_i \in A_i$, $\alpha_i(a_i)$ is the probability for playing a_i.

Terminology: When we talk about strategies in A_i specifically, to distinguish them from mixed strategies, we sometimes also call them pure strategies.

Definition (Mixed strategy profile)

A profile $\alpha = (\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$ of mixed strategies induces a probability distribution p_α over $A = \prod_{i \in N} A_i$ as follows:

$$p_\alpha(a) = \prod_{i \in N} \alpha_i(a_i).$$

For $A' \subseteq A$, we define

$$p_\alpha(A') = \sum_{a \in A'} p_\alpha(a) = \sum_{a \in A'} \prod_{i \in N} \alpha_i(a_i).$$

Notation

Since each pure strategy $a_i \in A_i$ is equivalent to its induced mixed strategy $\hat{a}_i$

$$\hat{a}_i(a_i') = \begin{cases} 1 & \text{if } a_i' = a_i \\ 0 & \text{otherwise,} \end{cases}$$

we sometimes abuse notation and write a_i instead of $\hat{a}_i$.

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Example (Mixed strategies for matching pennies)

	H	T
H	$1,-1$	$-1,\ 1$
T	$-1,\ 1$	$1,-1$

$\alpha = (\alpha_1, \alpha_2)$, $\alpha_1(H) = 2/3$, $\alpha_1(T) = 1/3$, $\alpha_2(H) = 1/3$, $\alpha_2(T) = 2/3$.

This induces a probability distribution over $\{H,T\} \times \{H,T\}$:

$$
\begin{aligned}
p_\alpha(H,H) &= \alpha_1(H) \cdot \alpha_2(H) = 2/9, & u_1(H,H) &= +1,\\
p_\alpha(H,T) &= \alpha_1(H) \cdot \alpha_2(T) = 4/9, & u_1(H,T) &= -1,\\
p_\alpha(T,H) &= \alpha_1(T) \cdot \alpha_2(H) = 1/9, & u_1(T,H) &= -1,\\
p_\alpha(T,T) &= \alpha_1(T) \cdot \alpha_2(T) = 2/9, & u_1(T,T) &= +1.
\end{aligned}
$$

Expected Utility

Definition (Expected utility)

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile.

The expected utility of α for player i is

$$U_i(\alpha) = U_i\left((\alpha_j)_{j \in N}\right) := \sum_{a \in A} p_\alpha(a)\, u_i(a) = \sum_{a \in A} \left(\prod_{j \in N} \alpha_j(a_j)\right) u_i(a).$$

Example (Mixed strategies for matching pennies (ctd.))

The expected utilities for player 1 and player 2 are

$$U_1(\alpha_1, \alpha_2) = -1/9 \qquad \text{and} \qquad U_2(\alpha_1, \alpha_2) = +1/9.$$

Expected Utility

Remark: The expected utility functions U_i are linear in all mixed strategies.

Proposition

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile, $\beta_i, \gamma_i \in \Delta(A_i)$ mixed strategies, and $\lambda \in [0,1]$. Then

$$U_i(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i) = \lambda U_i(\alpha_{-i}, \beta_i) + (1-\lambda) U_i(\alpha_{-i}, \gamma_i).$$

Moreover,

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) \cdot U_i(\alpha_{-i}, a_i)$$

Proof.

Homework. □

Definition (Mixed extension)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

The mixed extension of G is the game $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$ where

- $\Delta(A_i)$ is the set of probability distributions over A_i and
- $U_i : \prod_{j \in N} \Delta(A_j) \to \mathbb{R}$ assigns to each mixed strategy profile α the expected utility for player i according to the induced probability distribution p_α.

Nash Equilibria in Mixed Strategies

Definition (Nash equilibrium in mixed strategies)

Let G be a strategic game.

A Nash equilibrium in mixed strategies (or mixed-strategy Nash equilibrium) of G is a Nash equilibrium in the mixed extension of G.

Support

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Intuition:

- It does not make sense to assign positive probability to a pure strategy that is not a best response to what the other players do.
- Claim: A profile of mixed strategies α is a Nash equilibrium if and only if everyone only plays best pure responses to what the others play.

Definition (Support)

Let α_i be a mixed strategy.

The support of α_i is the set

$$supp(\alpha_i) = \{a_i \in A_i \mid \alpha_i(a_i) > 0\}$$

of actions played with nonzero probability.

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Lemma (Support lemma)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game.

Then $\alpha^* \in \prod_{i \in N} \Delta(A_i)$ is a mixed-strategy Nash equilibrium in G if and only if for every player $i \in N$, every pure strategy in the support of α_i^* is a best response to α_{-i}^*.

For a single player–given all other players stick to their mixed strategies–it does not make a difference whether he plays the mixed strategy or whether he plays any single pure strategy from the support of the mixed strategy.

Support Lemma

Example (Support lemma)

Matching pennies, strategy profile $\alpha = (\alpha_1, \alpha_2)$ with

$$\alpha_1(H) = {}^2\!/\!_3, \quad \alpha_1(T) = {}^1\!/\!_3, \quad \alpha_2(H) = {}^1\!/\!_3, \text{ and } \quad \alpha_2(T) = {}^2\!/\!_3.$$

For α to be a Nash equilibrium, both actions in $supp(\alpha_2) = \{H, T\}$ have to be best responses to α_1. Are they?

$$\begin{aligned} U_2(\alpha_1, H) &= \alpha_1(H) \cdot u_2(H, H) + \alpha_1(T) \cdot u_2(T, H) \\ &= {}^2\!/\!_3 \cdot (-1) + {}^1\!/\!_3 \cdot (+1) = -{}^1\!/\!_3, \\ U_2(\alpha_1, T) &= \alpha_1(H) \cdot u_2(H, T) + \alpha_1(T) \cdot u_2(T, T) \\ &= {}^2\!/\!_3 \cdot (+1) + {}^1\!/\!_3 \cdot (-1) = {}^1\!/\!_3. \end{aligned}$$

$\Rightarrow$ $H \in supp(\alpha_2)$, but $H \notin B_2(\alpha_1)$.

$\overset{\text{Support lemma}}{\Rightarrow}$ α can not be a Nash equilibrium.

Support Lemma

Proof.

"$\Rightarrow$": Let α^* be a Nash equilibrium with $a_i \in supp(\alpha_i^*)$.

Assume that a_i is not a best response to α_{-i}^*. Because U_i is linear, player i can improve his utility by shifting probability in α_i^* from a_i to a better response.

This makes the modified α_i^* a better response than the original α_i^*, i. e., the original α_i^* was not a best response, which contradicts the assumption that α^* is a Nash equilibrium.

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Support Lemma

Proof (ctd.)

"$\Leftarrow$": Assume that α^* is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy α'_i such that $U_i(\alpha^*_{-i}, \alpha'_i) > U_i(\alpha^*_{-i}, \alpha^*_i)$.

Because U_i is linear, there must be a pure strategy $a'_i \in supp(\alpha'_i)$ that has higher utility than some pure strategy $a''_i \in supp(\alpha^*_i)$.

Therefore, $supp(\alpha^*_i)$ does not only contain best responses to α^*_{-i}. □

Support Lemma

Proof (ctd.)

"$\Leftarrow$": Assume that α^* is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy α_i' such that $U_i(\alpha_{-i}^*, \alpha_i') > U_i(\alpha_{-i}^*, \alpha_i^*)$.

Because U_i is linear, there must be a pure strategy $a_i' \in supp(\alpha_i')$ that has higher utility than some pure strategy $a_i'' \in supp(\alpha_i^*)$.

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Therefore, $supp(\alpha_i^*)$ does not only contain best responses to α^*_{-i}. □

Computing Mixed-Strategy Nash Equilibria

Example (Mixed-strategy Nash equilibria in BoS)

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

We already know: (B,B) and (S,S) are pure Nash equilibria.

Possible supports (excluding "pure-vs-pure" strategies) are:

$\{B\}$ vs. $\{B,S\}$, $\{S\}$ vs. $\{B,S\}$, $\{B,S\}$ vs. $\{B\}$,
$\{B,S\}$ vs. $\{S\}$ and $\{B,S\}$ vs. $\{B,S\}$

Observation: In Bach or Stravinsky, pure strategies have unique best responses. Therefore, there can be no Nash equilibria of "pure-vs-strictly-mixed" type.

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Example (Mixed-strategy Nash equilibria in BoS (ctd.))

Consequence: Only need to search for additional Nash equilibria with support sets $\{B, S\}$ vs. $\{B, S\}$.
Assume that (α_1^*, α_2^*) is a Nash equilibrium with $0 < \alpha_1^*(B) < 1$ and $0 < \alpha_2^*(B) < 1$. Then

$$\begin{aligned}
& U_1(B, \alpha_2^*) = U_1(S, \alpha_2^*) \\
\Rightarrow\quad & 2 \cdot \alpha_2^*(B) + 0 \cdot \alpha_2^*(S) = 0 \cdot \alpha_2^*(B) + 1 \cdot \alpha_2^*(S) \\
\Rightarrow\quad & 2 \cdot \alpha_2^*(B) = 1 - \alpha_2^*(B) \\
\Rightarrow\quad & 3 \cdot \alpha_2^*(B) = 1 \\
\Rightarrow\quad & \alpha_2^*(B) = 1/3 \quad (\text{and } \alpha_2^*(S) = 2/3)
\end{aligned}$$

Similarly, we get $\alpha_1^*(B) = 2/3$ and $\alpha_1^*(S) = 1/3$.
The payoff profile of this equilibrium is $(2/3, 2/3)$.

Support Lemma

Remark

Let $G = \langle \{1,2\}, (A_i), (u_i) \rangle$ with $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$ be a two-player game with two actions each, and (T, α_2^*) with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of G.

Then at least one of the profiles (T, L) and (T, R) is also a Nash equilibrium of G.

Reason: Both L and R are best responses to T. Assume that T was neither a best response to L nor to R. Then B would be a better response than T both to L and to R.

With the linearity of U_1, B would also be a better response to α_2^* than T is. Contradiction.

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With the linearity of U_1, B would also be a better response to α_2^* than T is. Contradiction.

Support Lemma

Example

Consider the Nash equilibrium $\alpha^* = (\alpha_1^*, \alpha_2^*)$ with

$$\alpha_1^*(T) = 1, \quad \alpha_1^*(B) = 0, \quad \alpha_2^*(L) = {}^1\!/_{10}, \quad \alpha_2^*(R) = {}^9\!/_{10}$$

in the following game:

	L	R
T	1, 1	1, 1
B	2, 2	−5, −5

Here, (T, R) is also a Nash equilibrium.

Nash's Theorem

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Motivation: When does a strategic game have a mixed-strategy Nash equilibrium?

In the previous chapter, we discussed necessary and sufficient conditions for the existence of Nash equilibria for the special case of zero-sum games. Can we make other claims?

Nash's Theorem

Theorem (Nash's theorem)

Every finite strategic game has a mixed-strategy Nash equilibrium.

Proof sketch.

Consider the set-valued function of best responses $B : \mathbb{R}^{\sum_i |A_i|} \to 2^{\mathbb{R}^{\sum_i |A_i|}}$ with

$$B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i}).$$

A mixed strategy profile α is a fixed point of B if and only if $\alpha \in B(\alpha)$ if and only if α is a mixed-strategy Nash equilibrium.

The graph of B has to be connected. Then there is at least one point on the fixpoint diagonal. □

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Outline for the formal proof:

1. Review of necessary mathematical definitions
   ⇝ Subsection "Definitions"
2. Statement of a fixpoint theorem used to prove Nash's theorem (without proof)
   ⇝ Subsection "Kakutani's Fixpoint Theorem"
3. Proof of Nash's theorem using fixpoint theorem
   ⇝ Subsection "Proof of Nash's Theorem"

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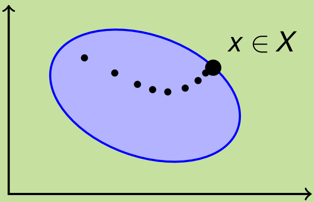
Definition

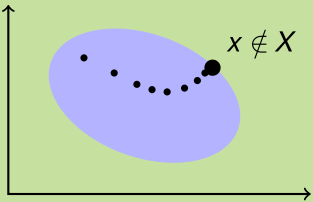
A set $X \subseteq \mathbb{R}^n$ is closed if X contains all its limit points, i. e., if $(x_k)_{k\in\mathbb{N}}$ is a sequence of elements in X and $\lim_{k\to\infty} x_k = x$, then also $x \in X$.

Example

Closed:



Not closed:



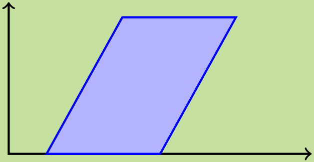
Definition

A set $X \subseteq \mathbb{R}^n$ is bounded if for each $i = 1, \ldots, n$ there are lower and upper bounds $a_i, b_i \in \mathbb{R}$ such that

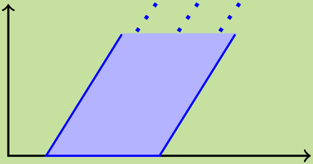
$$X \subseteq \prod_{i=1}^{n} [a_i, b_i].$$

Example

Bounded:



Not bounded:

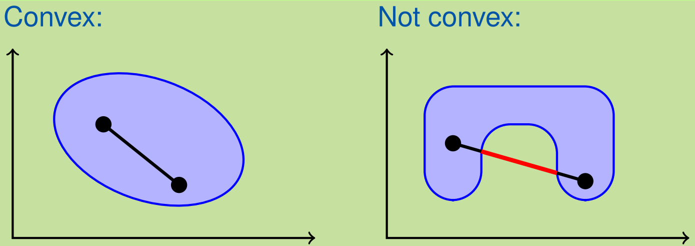


Definition

A set $X \subseteq \mathbb{R}^n$ is convex if for all $x, y \in X$ and all $\lambda \in [0,1]$,

$$\lambda x + (1-\lambda)y \in X.$$

Example



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Definition

For a function $f : X \to 2^X$, the graph of f is the set

$$Graph(f) = \{(x,y) \mid x \in X,\ y \in f(x)\}.$$

Theorem (Kakutani's fixpoint theorem)

Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed, bounded and convex set and let $f : X \to 2^X$ be a function such that

- *for all $x \in X$, the set $f(x) \subseteq X$ is nonempty and convex, and*
- *Graph(f) is closed.*

Then there is an $x \in X$ with $x \in f(x)$, i. e., f has a fixpoint.

Proof.

See Shizuo Kakutani, A generalization of Brouwer's fixed point theorem, 1941, or your favorite advanced calculus textbook, or the Internet.

For German speakers: Harro Heuser, Lehrbuch der Analysis, Teil 2, also has a proof (Abschnitt 232). □

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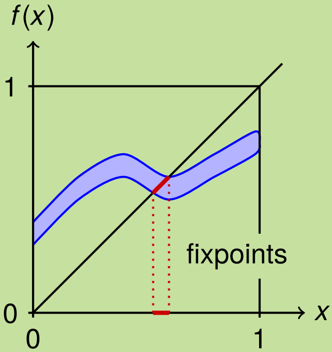
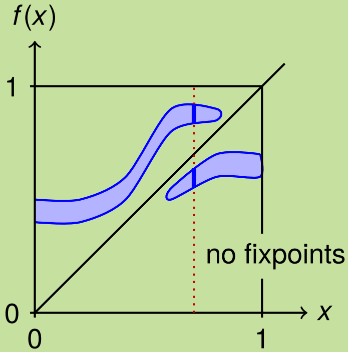
Example

Let $X = [0,1]$.

Kakutani's theorem applicable:



Kakutani's theorem not applicable:



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Proof.

Apply Kakutani's fixpoint theorem using $X = \mathscr{A} = \prod_{i \in N} \Delta(A_i)$ and $f = B$, where $B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i})$.

We have to show:

1. $\mathscr{A}$ is nonempty,
2. $\mathscr{A}$ is closed,
3. $\mathscr{A}$ is bounded,
4. $\mathscr{A}$ is convex,
5. $B(\alpha)$ is nonempty for all $\alpha \in \mathscr{A}$,
6. $B(\alpha)$ is convex for all $\alpha \in \mathscr{A}$, and
7. $\mathit{Graph}(B)$ is closed.

Proof (ctd.)

Some notation:

- Assume without loss of generality that $N = \{1, \ldots, n\}$.
- A profile of mixed strategies can be written as a vector of $M = \sum_{i \in N} |A_i|$ real numbers in the interval $[0, 1]$ such that numbers for the same player add up to 1.

  For example, $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1(T) = 0.7$, $\alpha_1(M) = 0.0$, $\alpha_1(B) = 0.3$, $\alpha_2(L) = 0.4$, $\alpha_2(R) = 0.6$ can be seen as the vector

$$(\underbrace{0.7,\ 0.0,\ 0.3}_{\alpha_1},\ \underbrace{0.4,\ 0.6}_{\alpha_2})$$

- This allows us to interpret the set $\mathscr{A}$ of mixed strategy profiles as a subset of $\mathbb{R}^M$.

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Proof (ctd.)

1. $\mathscr{A}$ nonempty: Trivial. $\mathscr{A}$ contains the tuple

$$(1, \underbrace{0,\dots,0}_{|A_1|-1 \text{ times}}, \dots, 1, \underbrace{0,\dots,0}_{|A_n|-1 \text{ times}}).$$

2. $\mathscr{A}$ closed: Let $\alpha_1, \alpha_2, \dots$ be a sequence in $\mathscr{A}$ that converges to $\lim_{k\to\infty} \alpha_k = \alpha$. Suppose $\alpha \notin \mathscr{A}$. Then either there is some component of α that is less than zero or greater than one, or the components for some player i add up to a value other than one.

   Since α is a limit point, the same must hold for some α_k in the sequence. But then, $\alpha_k \notin \mathscr{A}$, a contradiction. Hence $\mathscr{A}$ is closed.

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Proof (ctd.)

3. $\mathscr{A}$ **bounded:** Trivial. All entries are between 0 and 1, i. e., $\mathscr{A}$ is bounded by $[0,1]^M$.
4. $\mathscr{A}$ **convex:** Let $\alpha, \beta \in \mathscr{A}$ and $\lambda \in [0,1]$, and consider $\gamma = \lambda\alpha + (1-\lambda)\beta$. Then

$$\begin{aligned}\min(\gamma) &= \min(\lambda\alpha + (1-\lambda)\beta)\\ &\geq \lambda \cdot \min(\alpha) + (1-\lambda)\cdot\min(\beta)\\ &\geq \lambda \cdot 0 + (1-\lambda)\cdot 0 = 0,\end{aligned}$$

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4 $\mathscr{A}$ convex (ctd.): Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be the sections of α, β and γ, respectively, that determine the probability distribution for player *i*. Then

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$$U_i\big(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i\big) = \lambda U_i\big(\alpha_{-i}, \beta_i\big) + (1-\lambda)U_i\big(\alpha_{-i}, \gamma_i\big) \quad (1)$$

for all $\lambda \in [0, 1]$.

Hence, U_i is continous on $\Delta(A_i)$.

Continous functions on closed and bounded sets take their maximum in that set.

Therefore, $B_i(\alpha_{-i}) \neq \emptyset$ for all $i \in N$, and thus $B(\alpha) \neq \emptyset$.

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   Then $U_i(\alpha_{-i}, \alpha_i') = U_i(\alpha_{-i}, \alpha_i'')$.

   With Equation (1), this implies

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Proof (ctd.)

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$$
\begin{aligned}
U_i\left(\alpha_{-i}, \beta_i\right) &\overset{(D)}{=} U_i\left(\lim_{k\to\infty}(\alpha^k_{-i}, \beta^k_i)\right) \\
&\overset{(C)}{=} \lim_{k\to\infty} U_i\left(\alpha^k_{-i}, \beta^k_i\right) \\
&\overset{(B)}{\geq} \lim_{k\to\infty} U_i\left(\alpha^k_{-i}, \beta'_i\right) \quad \text{for all } \beta'_i \in \Delta(A_i) \\
&\overset{(C)}{=} U_i\left(\lim_{k\to\infty} \alpha^k_{-i}, \beta'_i\right) \quad \text{for all } \beta'_i \in \Delta(A_i) \\
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\end{aligned}
$$

(D): def. α_i, β_i; (C) continuity; (B) β^k_i best response to α^k_{-i}.

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Correlated Equilibria

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Recall: There are three Nash equilibria in Bach or Stravinsky

- (B,B) with payoff profile $(2,1)$
- (S,S) with payoff profile $(1,2)$
- (α_1^*, α_2^*) with payoff profile $(2/3, 2/3)$ where
  - $\alpha_1^*(B) = 2/3$, $\alpha_1^*(S) = 1/3$,
  - $\alpha_2^*(B) = 1/3$, $\alpha_2^*(S) = 2/3$.

Idea: Use a publicly visible coin toss to decide which action from a mixed strategy is played. This can lead to higher payoffs.

Correlated Equilibria

Example (Correlated equilibrium in BoS)

With a fair coin that both players can observe, the players can agree to play as follows:

- If the coin shows heads, both play B.
- If the coin shows tails, both play S.

This is stable in the sense that no player has an incentive to deviate from this agreed-upon rule, as long as the other player keeps playing his/her strategy (cf. definition of Nash equilibria).

Expected payoffs: $(3/2, 3/2)$ instead of $(2/3, 2/3)$.

Observations and Information Partitions

We assume that observations are made based on a finite probability space (Ω, π), where Ω is a set of states and π is a probability measure on Ω.

Agents might not be able to distingush all states from each other. In order to model this, we assume for each player i an information partition $\mathscr{P}_i = \{P_{i1}, P_{i2}, \dots, P_{ik_i}\}$. This means that $\bigcup \mathscr{P}_i = \Omega$ for all i, and for all $P_j, P_k \in \mathscr{P}_i$ with $P_j \neq P_k$, we have $P_j \cap P_k = \emptyset$.

Example: $\Omega = \{x, y, z\}$, $\mathscr{P}_1 = \{\{x\}, \{y, z\}\}$, $\mathscr{P}_2 = \{\{x, y\}, \{z\}\}$.

We say that a function $f : \Omega \to X$ respects an information partition for player i if $f(\omega) = f(\omega')$ whenever $\omega, \omega' \in P_i$ for some $P_i \in \mathscr{P}_i$.

Example: f respects $\mathscr{P}_1$ if $f(y) = f(z)$.

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Correlated Equilibria – Formally

Definition

A correlated equilibrium of a strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ consists of

- a finite probability space (Ω, π),
- for each player $i \in N$ an information partition $\mathscr{P}_i$ of Ω,
- for each player $i \in N$ a function $\sigma_i : \Omega \to A_i$ that respects $\mathscr{P}_i$ (σ_i is player i's strategy)

such that for every $i \in N$ and every function $\tau_i : \Omega \to A_i$ that respects $\mathscr{P}_i$ (i.e. for every possible strategy of player i) we have

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)). \quad (2)$$

Example

	L	R
T	$6,6$	$2,7$
B	$7,2$	$0,0$

Equilibria: (T,R) with $(2,7)$, (B,L) with $(7,2)$, and mixed $((\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}))$ with $(4+\frac{2}{3},4+\frac{2}{3})$.

Assume $\Omega = \{x,y,z\}, \pi(x) = \frac{1}{3}, \pi(y) = \frac{1}{3}, \pi(z) = \frac{1}{3}$.
Assume further $\mathscr{P}_1 = \{\{x\},\{y,z\}\}, \mathscr{P}_2 = \{\{x,y\},\{z\}\}$.
Set $\sigma_1(x) = B, \sigma_1(y) = \sigma_1(z) = T$ and $\sigma_2(x) = \sigma_2(y) = L, \sigma_2(z) = R$.

Then both player play optimally and get a payoff profile of $(5,5)$.

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Assume $\Omega = \{x,y,z\}, \pi(x) = \frac{1}{3}, \pi(y) = \frac{1}{3}, \pi(z) = \frac{1}{3}$.
Assume further $\mathscr{P}_1 = \{\{x\},\{y,z\}\}, \mathscr{P}_2 = \{\{x,y\},\{z\}\}$.
Set $\sigma_1(x) = B, \sigma_1(y) = \sigma_1(z) = T$ and $\sigma_2(x) = \sigma_2(y) = L, \sigma_2(z) = R$.

Then both player play optimally and get a payoff profile of $(5,5)$.

Example

	L	R
T	$6,6$	$2,7$
B	$7,2$	$0,0$

Equilibria: (T,R) with $(2,7)$, (B,L) with $(7,2)$, and mixed $((\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}))$ with $(4+\frac{2}{3},4+\frac{2}{3})$.

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Connection to Nash Equilibria

Proposition

For every mixed strategy Nash equilibrium α of a finite strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, there is a correlated equilibrium $\langle (\Omega, \pi), (\mathscr{P}_i), (\sigma_i) \rangle$ in which for each player i the distribution on A_i induced by σ_i is α_i.

This means that correlated equilibria are a generalization of Nash equilibria.

Proof

Proof.

Let $\Omega = A$ and define $\pi(a) = \prod_{j \in N} \alpha_j(a_j)$. For each player i, let $a \in P$ and $b \in P$ for $P \in \mathscr{P}_i$ if $a_i = b_i$. Define $\sigma_i(a) = a_i$ for each $a \in A$.

Then $\langle(\Omega, \pi), (\mathscr{P}_i), (\sigma_i)\rangle$ is a correlated equilibrium since the left hand side of (2) is the Nash equilibrium payoff and for each player i at least as good any other strategy τ_i respecting the information partition. Further, the distribution induced by σ_i is α_i. □

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Proposition

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. Any convex combination of correlated equilibirum payoff profiles of G is a correlated equilibirum payoff profile of G.

Proof idea: From given equilibria and weighting factors, create a new one by combining them orthogonally, using the weighting factors.

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Let $u^1, \ldots, u^K$ be the payoff profiles and let $(\lambda^1, \ldots, \lambda^K) \in \mathbb{R}^K$ with $\lambda^l \geq 0$ and $\sum_{l=1}^K \lambda^l = 1$. For each l let $\langle (\Omega^l, \pi^l), (\mathscr{P}_i^l), (\sigma_i^l) \rangle$ be a correlated equilibrium generating payoff u^l. Wlog. assume all Ω^l's are disjoint.

Now we define a correlated equilibrium generating the payoff $\sum_{l=1}^K \lambda^l u^l$. Let $\Omega = \bigcup_l \Omega^l$. For any $\omega \in \Omega$ define $\pi(\omega) = \lambda^l \pi^l(\omega)$ where l is such that $\omega \in \Omega^l$. For each $i \in N$ let $\mathscr{P}_i = \bigcup_l \mathscr{P}_i^l$ and set $\sigma_i(\omega) = \sigma_i^l(\omega)$ where l is such that $\omega \in \Omega^l$. □

Basically, first throw a dice for which CE to go for, then proceed in this CE.

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Mixed Strategies
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- Mixed strategies allow randomization.
- Characterization of mixed-strategy Nash equilibria: players only play best responses with positive probability (support lemma).
- Nash's Theorem: Every finite strategic game has a mixed-strategy Nash equilibrium.
- Correlated equilibria can lead to higher payoffs.
- All Nash equilibria are correlated equilibria, but not *vice versa*.