Game Theory

3. Mixed Strategies

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1 Mixed Strategies

- Definitions
- Support Lemma
Observation: Not every strategic game has a pure-strategy Nash equilibrium (e.g. matching pennies).

Question:
- Can we do anything about that?
- Which strategy to play then?

Idea: Consider randomized strategies.
Mixed Strategies

Notation

Let $X$ be a set.

Then $\Delta(X)$ denotes the set of probability distributions over $X$.

That is, each $p \in \Delta(X)$ is a mapping $p : X \to [0, 1]$ with

$$\sum_{x \in X} p(x) = 1.$$
Mixed Strategies

A mixed strategy is a strategy where a player is allowed to randomize his action (throw a dice mentally and then act according to what he has decided to do for each outcome).

**Definition (Mixed strategy)**

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A mixed strategy of player $i$ in $G$ is a probability distribution $\alpha_i \in \Delta(A_i)$ over player $i$’s actions.

For $a_i \in A_i$, $\alpha_i(a_i)$ is the probability for playing $a_i$.

Terminology: When we talk about strategies in $A_i$ specifically, to distinguish them from mixed strategies, we sometimes also call them pure strategies.
**Definition (Mixed strategy profile)**

A profile $\alpha = (\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$ of mixed strategies induces a probability distribution $p_\alpha$ over $A = \prod_{i \in N} A_i$ as follows:

$$p_\alpha(a) = \prod_{i \in N} \alpha_i(a_i).$$

For $A' \subseteq A$, we define

$$p_\alpha(A') = \sum_{a \in A'} p_\alpha(a) = \sum_{a \in A'} \prod_{i \in N} \alpha_i(a_i).$$
Mixed Strategies

Notation

Since each pure strategy \( a_i \in A_i \) is equivalent to its induced mixed strategy \( \hat{a}_i \)

\[
\hat{a}_i(a'_i) = \begin{cases} 
1 & \text{if } a'_i = a_i \\
0 & \text{otherwise,}
\end{cases}
\]

we sometimes abuse notation and write \( a_i \) instead of \( \hat{a}_i \).
Mixed Strategies

Example (Mixed strategies for matching pennies)

\[ \begin{array}{c|cc}
 & H & T \\
\hline
H & 1, -1 & -1, 1 \\
T & -1, 1 & 1, -1 \\
\end{array} \]

\[ \alpha = (\alpha_1, \alpha_2), \quad \alpha_1(H) = \frac{2}{3}, \quad \alpha_1(T) = \frac{1}{3}, \quad \alpha_2(H) = \frac{1}{3}, \quad \alpha_2(T) = \frac{2}{3}. \]

This induces a probability distribution over \( \{H, T\} \times \{H, T\} \):

\[ p_\alpha(H, H) = \alpha_1(H) \cdot \alpha_2(H) = \frac{2}{9}, \quad u_1(H, H) = +1, \]
\[ p_\alpha(H, T) = \alpha_1(H) \cdot \alpha_2(T) = \frac{4}{9}, \quad u_1(H, T) = -1, \]
\[ p_\alpha(T, H) = \alpha_1(T) \cdot \alpha_2(H) = \frac{1}{9}, \quad u_1(T, H) = -1, \]
\[ p_\alpha(T, T) = \alpha_1(T) \cdot \alpha_2(T) = \frac{2}{9}, \quad u_1(T, T) = +1. \]
Expected Utility

Definition (Expected utility)
Let \( \alpha \in \prod_{i \in N} \Delta(A_i) \) be a mixed strategy profile. The expected utility of \( \alpha \) for player \( i \) is

\[
U_i(\alpha) = U_i((\alpha_j)_{j \in N}) := \sum_{a \in A} p_\alpha(a) \ u_i(a) = \sum_{a \in A} \left( \prod_{j \in N} \alpha_j(a_j) \right) u_i(a).
\]

Example (Mixed strategies for matching pennies (ctd.))
The expected utilities for player 1 and player 2 are

\[
U_1(\alpha_1, \alpha_2) = -\frac{1}{9} \quad \text{and} \quad U_2(\alpha_1, \alpha_2) = \frac{1}{9}.
\]
Expected Utility

**Remark:** The expected utility functions $U_i$ are linear in all mixed strategies.

**Proposition**

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile, $\beta_i, \gamma_i \in \Delta(A_i)$ mixed strategies, and $\lambda \in [0,1]$. Then

$$U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i) = \lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i).$$

Moreover,

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) \cdot U_i(\alpha_{-i}, a_i)$$

**Proof.**

Homework.
Definition (Mixed extension)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

The **mixed extension** of $G$ is the game $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$ where

- $\Delta(A_i)$ is the set of probability distributions over $A_i$ and
- $U_i : \prod_{j \in N} \Delta(A_j) \to \mathbb{R}$ assigns to each mixed strategy profile $\alpha$ the expected utility for player $i$ according to the induced probability distribution $p_\alpha$. 


Definition (Nash equilibrium in mixed strategies)

Let $G$ be a strategic game.

A Nash equilibrium in mixed strategies (or mixed-strategy Nash equilibrium) of $G$ is a Nash equilibrium in the mixed extension of $G$. 
Intuition:

- It does not make sense to assign positive probability to a pure strategy that is not a best response to what the other players do.
- **Claim:** A profile of mixed strategies $\alpha$ is a Nash equilibrium if and only if everyone only plays best pure responses to what the others play.

**Definition (Support)**

Let $\alpha_i$ be a mixed strategy. The **support** of $\alpha_i$ is the set

$$supp(\alpha_i) = \{a_i \in A_i \mid \alpha_i(a_i) > 0\}$$

of actions played with nonzero probability.
Support Lemma

Lemma (Support lemma)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game.

Then $\alpha^* \in \prod_{i \in N} \Delta(A_i)$ is a mixed-strategy Nash equilibrium in $G$ if and only if for every player $i \in N$, every pure strategy in the support of $\alpha^*_i$ is a best response to $\alpha^*_{-i}$.

For a single player—given all other players stick to their mixed strategies—it does not make a difference whether he plays the mixed strategy or whether he plays any single pure strategy from the support of the mixed strategy.
Example (Support lemma)

Matching pennies, strategy profile $\alpha = (\alpha_1, \alpha_2)$ with

$$\alpha_1(H) = \frac{2}{3}, \quad \alpha_1(T) = \frac{1}{3}, \quad \alpha_2(H) = \frac{1}{3}, \quad \text{and} \quad \alpha_2(T) = \frac{2}{3}.$$ 

For $\alpha$ to be a Nash equilibrium, both actions in $\text{supp}(\alpha_2) = \{H, T\}$ have to be best responses to $\alpha_1$. Are they?

$$U_2(\alpha_1, H) = \alpha_1(H) \cdot u_2(H, H) + \alpha_1(T) \cdot u_2(T, H)$$
$$= \frac{2}{3} \cdot (-1) + \frac{1}{3} \cdot (+1) = -\frac{1}{3},$$

$$U_2(\alpha_1, T) = \alpha_1(H) \cdot u_2(H, T) + \alpha_1(T) \cdot u_2(T, T)$$
$$= \frac{2}{3} \cdot (+1) + \frac{1}{3} \cdot (-1) = \frac{1}{3}.$$ 

$H \in \text{supp}(\alpha_2)$, but $H \notin B_2(\alpha_1)$.

Support lemma $\Rightarrow \alpha$ can not be a Nash equilibrium.
Support Lemma

Proof.

“⇒”: Let $\alpha^*$ be a Nash equilibrium with $a_i \in \text{supp}(\alpha_i^*)$. Assume that $a_i$ is not a best response to $\alpha_{-i}^*$. Because $U_i$ is linear, player $i$ can improve his utility by shifting probability in $\alpha_i^*$ from $a_i$ to a better response. This makes the modified $\alpha_i^*$ a better response than the original $\alpha_i^*$, i.e., the original $\alpha_i^*$ was not a best response, which contradicts the assumption that $\alpha^*$ is a Nash equilibrium.
Support Lemma

Proof (ctd.)

“⇐”: Assume that $\alpha^*$ is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy $\alpha'_i$ such that

$U_i(\alpha^*_{-i}, \alpha'_i) > U_i(\alpha^*_{-i}, \alpha^*_i)$.

Because $U_i$ is linear, there must be a pure strategy $a'_i \in supp(\alpha'_i)$ that has higher utility than some pure strategy $a''_i \in supp(\alpha^*_i)$.

Therefore, $supp(\alpha^*_i)$ does not only contain best responses to $\alpha^*_{-i}$. 

$\square$
Computing Mixed-Strategy Nash Equilibria

Example (Mixed-strategy Nash equilibria in BoS)

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>S</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

We already know: \((B, B)\) and \((S, S)\) are pure Nash equilibria. Possible supports (excluding “pure-vs-pure” strategies) are:

\[
\{B\} \text{ vs. } \{B, S\}, \quad \{S\} \text{ vs. } \{B, S\}, \quad \{B, S\} \text{ vs. } \{B\}, \quad \{B, S\} \text{ vs. } \{S\} \quad \text{and} \quad \{B, S\} \text{ vs. } \{B, S\}
\]

Observation: In Bach or Stravinsky, pure strategies have unique best responses. Therefore, there can be no Nash equilibria of “pure-vs-strictly-mixed” type.
Computing Mixed-Strategy Nash Equilibria

Example (Mixed-strategy Nash equilibria in BoS (ctd.))

**Consequence:** Only need to search for additional Nash equilibria with support sets \{B, S\} vs. \{B, S\}.

Assume that \((\alpha_1^*, \alpha_2^*)\) is a Nash equilibrium with \(0 < \alpha_1^*(B) < 1\) and \(0 < \alpha_2^*(B) < 1\). Then

\[
U_1(B, \alpha_2^*) = U_1(S, \alpha_2^*)
\]

\[
\Rightarrow 2 \cdot \alpha_2^*(B) + 0 \cdot \alpha_2^*(S) = 0 \cdot \alpha_2^*(B) + 1 \cdot \alpha_2^*(S)
\]

\[
\Rightarrow 2 \cdot \alpha_2^*(B) = 1 - \alpha_2^*(B)
\]

\[
\Rightarrow 3 \cdot \alpha_2^*(B) = 1
\]

\[
\Rightarrow \alpha_2^*(B) = \frac{1}{3} \quad \text{(and } \alpha_2^*(S) = \frac{2}{3})
\]

Similarly, we get \(\alpha_1^*(B) = \frac{2}{3}\) and \(\alpha_1^*(S) = \frac{1}{3}\).

The payoff profile of this equilibrium is \((\frac{2}{3}, \frac{2}{3})\).
Support Lemma

Remark

Let $G = \langle \{1, 2\}, (A_i), (u_i) \rangle$ with $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$ be a two-player game with two actions each, and $(T, \alpha_2^*)$ with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of $G$.

Then at least one of the profiles $(T, L)$ and $(T, R)$ is also a Nash equilibrium of $G$.

**Reason:** Both $L$ and $R$ are best responses to $T$. Assume that $T$ was neither a best response to $L$ nor to $R$. Then $B$ would be a better response than $T$ both to $L$ and to $R$.

With the linearity of $U_1$, $B$ would also be a better response to $\alpha_2^*$ than $T$ is. Contradiction.
Support Lemma

Example

Consider the Nash equilibrium $\alpha^* = (\alpha_1^*, \alpha_2^*)$ with

$$\alpha_1^*(T) = 1, \quad \alpha_1^*(B) = 0, \quad \alpha_2^*(L) = 1/10, \quad \alpha_2^*(R) = 9/10$$

in the following game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>1, 1</td>
<td>1, 1</td>
</tr>
<tr>
<td>B</td>
<td>2, 2</td>
<td>−5, −5</td>
</tr>
</tbody>
</table>

Here, $(T, R)$ is also a Nash equilibrium.
2 Nash’s Theorem

- Definitions
- Kakutani’s Fixpoint Theorem
- Proof of Nash’s Theorem
Motivation: When does a strategic game have a mixed-strategy Nash equilibrium?

In the previous chapter, we discussed necessary and sufficient conditions for the existence of Nash equilibria for the special case of zero-sum games. Can we make other claims?
Theorem (Nash’s theorem)

Every finite strategic game has a mixed-strategy Nash equilibrium.

Proof sketch.

Consider the set-valued function of best responses

\[ B : \mathbb{R}^{\sum_i |A_i|} \rightarrow 2^{\mathbb{R}^{\sum_i |A_i|}} \]

with

\[ B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i}) \]

A mixed strategy profile \( \alpha \) is a fixed point of \( B \) if and only if \( \alpha \in B(\alpha) \) if and only if \( \alpha \) is a mixed-strategy Nash equilibrium.

The graph of \( B \) has to be connected. Then there is at least one point on the fixpoint diagonal.
Nash’s Theorem

Outline for the formal proof:

1. Review of necessary mathematical definitions
   ⇔ Subsection “Definitions”

2. Statement of a fixpoint theorem used to prove Nash’s theorem (without proof)
   ⇔ Subsection “Kakutani’s Fixpoint Theorem”

3. Proof of Nash’s theorem using fixpoint theorem
   ⇔ Subsection “Proof of Nash’s Theorem”
Nash’s Theorem
Definitions

Definition

A set \( X \subseteq \mathbb{R}^n \) is closed if \( X \) contains all its limit points, i.e., if \((x_k)_{k\in\mathbb{N}}\) is a sequence of elements in \( X \) and \( \lim_{k \to \infty} x_k = x \), then also \( x \in X \).

Example

Closed:

Not closed:
Nash’s Theorem

Definitions

Definition

A set $X \subseteq \mathbb{R}^n$ is **bounded** if for each $i = 1, \ldots, n$ there are lower and upper bounds $a_i, b_i \in \mathbb{R}$ such that

$$X \subseteq \prod_{i=1}^{n} [a_i, b_i].$$

Example

**Bounded:**

![Bounded example](image1)

**Not bounded:**

![Not bounded example](image2)
Nash’s Theorem

Definitions

Definition
A set $X \subseteq \mathbb{R}^n$ is convex if for all $x, y \in X$ and all $\lambda \in [0, 1],$

$$\lambda x + (1 - \lambda)y \in X.$$ 

Example
Convex:

Not convex:
Nash’s Theorem

Definitions

Definition

For a function $f : X \rightarrow 2^X$, the graph of $f$ is the set

$$\text{Graph}(f) = \{(x, y) \mid x \in X, y \in f(x)\}.$$
Theorem (Kakutani’s fixpoint theorem)

Let \( X \subseteq \mathbb{R}^n \) be a nonempty, closed, bounded and convex set and let \( f : X \rightarrow 2^X \) be a function such that

- for all \( x \in X \), the set \( f(x) \subseteq X \) is nonempty and convex, and
- \( \text{Graph}(f) \) is closed.

Then there is an \( x \in X \) with \( x \in f(x) \), i.e., \( f \) has a fixpoint.

Proof.

See Shizuo Kakutani, A generalization of Brouwer’s fixed point theorem, 1941, or your favorite advanced calculus textbook, or the Internet.

For German speakers: Harro Heuser, Lehrbuch der Analysis, Teil 2, also has a proof (Abschnitt 232).
Nash’s Theorem
Kakutani’s Fixpoint Theorem

Example
Let $X = [0, 1]$.

Kakutani’s theorem applicable:

Kakutani’s theorem not applicable:
Proof.

Apply Kakutani’s fixpoint theorem using \( X = \mathcal{A} = \prod_{i \in N} \Delta(A_i) \) and \( f = B \), where \( B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i}) \).

We have to show:

1. \( \mathcal{A} \) is nonempty,
2. \( \mathcal{A} \) is closed,
3. \( \mathcal{A} \) is bounded,
4. \( \mathcal{A} \) is convex,
5. \( B(\alpha) \) is nonempty for all \( \alpha \in \mathcal{A} \),
6. \( B(\alpha) \) is convex for all \( \alpha \in \mathcal{A} \), and
7. \( \text{Graph}(B) \) is closed.
Nash’s Theorem

Proof

Proof (ctd.)

Some notation:

■ Assume without loss of generality that \( N = \{1, \ldots, n\} \).

■ A profile of mixed strategies can be written as a vector of \( M = \sum_{i \in N} |A_i| \) real numbers in the interval \([0, 1]\) such that numbers for the same player add up to 1.

For example, \( \alpha = (\alpha_1, \alpha_2) \) with \( \alpha_1(T) = 0.7, \alpha_1(M) = 0.0, \alpha_1(B) = 0.3, \alpha_2(L) = 0.4, \alpha_2(R) = 0.6 \) can be seen as the vector

\[
\begin{pmatrix}
0.7, & 0.0, & 0.3, & 0.4, & 0.6 \\
\alpha_1, & \alpha_2
\end{pmatrix}
\]

■ This allows us to interpret the set \( \mathcal{A} \) of mixed strategy profiles as a subset of \( \mathbb{R}^M \).
Proof (ctd.)

1. \( \mathcal{A} \) nonempty: Trivial. \( \mathcal{A} \) contains the tuple

\[
(1, 0, \ldots, 0, \ldots, 1, 0, \ldots, 0),
\]

where \( |A_1| - 1 \) times \( \ldots \), where \( |A_n| - 1 \) times.

2. \( \mathcal{A} \) closed: Let \( \alpha_1, \alpha_2, \ldots \) be a sequence in \( \mathcal{A} \) that converges to \( \lim_{k \to \infty} \alpha_k = \alpha \). Suppose \( \alpha \notin \mathcal{A} \). Then either there is some component of \( \alpha \) that is less than zero or greater than one, or the components for some player \( i \) add up to a value other than one.

Since \( \alpha \) is a limit point, the same must hold for some \( \alpha_k \) in the sequence. But then, \( \alpha_k \notin \mathcal{A} \), a contradiction. Hence \( \mathcal{A} \) is closed.
Proof (ctd.)

3. \( \mathcal{A} \) bounded: Trivial. All entries are between 0 and 1, i.e., \( \mathcal{A} \) is bounded by \([0, 1]^M\).

4. \( \mathcal{A} \) convex: Let \( \alpha, \beta \in \mathcal{A} \) and \( \lambda \in [0, 1] \), and consider \( \gamma = \lambda \alpha + (1 - \lambda) \beta \). Then

\[
\min(\gamma) = \min(\lambda \alpha + (1 - \lambda) \beta) \\
\geq \lambda \cdot \min(\alpha) + (1 - \lambda) \cdot \min(\beta) \\
\geq \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0,
\]

and similarly, \( \max(\gamma) \leq 1 \).

Hence, all entries in \( \gamma \) are still in \([0, 1]\).
Proof (ctd.)

4. \( \mathcal{A} \) convex (ctd.): Let \( \tilde{\alpha}, \tilde{\beta} \) and \( \tilde{\gamma} \) be the sections of \( \alpha, \beta \) and \( \gamma \), respectively, that determine the probability distribution for player \( i \). Then

\[
\sum \tilde{\gamma} = \sum (\lambda \tilde{\alpha} + (1 - \lambda) \tilde{\beta}) \\
= \lambda \cdot \sum \tilde{\alpha} + (1 - \lambda) \cdot \sum \tilde{\beta} \\
= \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1.
\]

Hence, all probabilities for player \( i \) in \( \gamma \) still sum up to 1. Altogether, \( \gamma \in \mathcal{A} \), and therefore, \( \mathcal{A} \) is convex.
Proof (ctd.)

5. \( B(\alpha) \) nonempty: For a fixed \( \alpha_{-i} \), \( U_i \) is linear in the mixed strategies of player \( i \), i.e., for \( \beta_i, \gamma_i \in \Delta(A_i) \),

\[
U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i) = \lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i)
\]

for all \( \lambda \in [0,1] \).

Hence, \( U_i \) is continuous on \( \Delta(A_i) \).

Continuous functions on closed and bounded sets take their maximum in that set.

Therefore, \( B_i(\alpha_{-i}) \neq \emptyset \) for all \( i \in N \), and thus \( B(\alpha) \neq \emptyset \).
Proof (ctd.)

6 \( B(\alpha) \) convex: This follows, since each \( B_i(\alpha_{-i}) \) is convex.

To see this, let \( \alpha'_i, \alpha''_i \in B_i(\alpha_{-i}) \).

Then \( U_i(\alpha_{-i}, \alpha'_i) = U_i(\alpha_{-i}, \alpha''_i) \).

With Equation (1), this implies

\[
\lambda \alpha'_i + (1 - \lambda) \alpha''_i \in B_i(\alpha_{-i}).
\]

Hence, \( B_i(\alpha_{-i}) \) is convex.

7 Graph\((B)\) closed: Let \( (\alpha^k, \beta^k) \) be a convergent sequence in Graph\((B)\) with \( \lim_{k \to \infty} (\alpha^k, \beta^k) = (\alpha, \beta) \).

So, \( \alpha^k, \beta^k, \alpha, \beta \in \prod_{i \in N} \Delta(A_i) \) and \( \beta^k \in B(\alpha^k) \).

We need to show that \( (\alpha, \beta) \in \text{Graph}(B) \), i.e., that \( \beta \in B(\alpha) \).
Proof (ctd.)

**Graph(B) closed (ctd.):** It holds for all \( i \in N \):

\[
U_i(\alpha_{-i}, \beta_i) \overset{(D)}{=} U_i\left( \lim_{k \to \infty} (\alpha^k_{-i}, \beta^k_i) \right)
\]

\[
\overset{(C)}{=} \lim_{k \to \infty} U_i(\alpha^k_{-i}, \beta^k_i)
\]

\[
\overset{(B)}{\geq} \lim_{k \to \infty} U_i(\alpha^k_{-i}, \beta'_i) \quad \text{for all } \beta'_i \in \Delta(A_i)
\]

\[
\overset{(C)}{=} U_i\left( \lim_{k \to \infty} \alpha^k_{-i}, \beta'_i \right) \quad \text{for all } \beta'_i \in \Delta(A_i)
\]

\[
\overset{(D)}{=} U_i(\alpha_{-i}, \beta'_i) \quad \text{for all } \beta'_i \in \Delta(A_i).
\]

(D): def. \( \alpha_i, \beta_i \); (C) continuity; (B) \( \beta^k_i \) best response to \( \alpha^k_{-i} \).
Nash’s Theorem
Proof

Proof (ctd.)

7. **Graph**(B) closed (ctd.): It follows that \( \beta_i \) is a best response to \( \alpha_{-i} \) for all \( i \in N \).

Thus, \( \beta \in B(\alpha) \) and finally \( (\alpha, \beta) \in \text{Graph}(B) \).

Therefore, all requirements of Kakutani’s fixpoint theorem are satisfied.

Applying Kakutani’s theorem establishes the existence of a fixpoint of \( B \), which is, by definition/construction, the same as a mixed-strategy Nash equilibrium.  

\[ \square \]
3 Correlated Equilibria
Correlated Equilibria

Recall: There are three Nash equilibria in Bach or Stravinsky

- \((B, B)\) with payoff profile \((2,1)\)
- \((S, S)\) with payoff profile \((1,2)\)
- \((\alpha_1^*, \alpha_2^*)\) with payoff profile \((2/3, 2/3)\) where
  - \(\alpha_1^*(B) = 2/3, \alpha_1^*(S) = 1/3,\)
  - \(\alpha_2^*(B) = 1/3, \alpha_2^*(S) = 2/3.\)

Idea: Use a publicly visible coin toss to decide which action from a mixed strategy is played. This can lead to higher payoffs.
Correlated Equilibria

Example (Correlated equilibrium in BoS)

With a fair coin that both players can observe, the players can agree to play as follows:

- If the coin shows heads, both play $B$.
- If the coin shows tails, both play $S$.

This is stable in the sense that no player has an incentive to deviate from this agreed-upon rule, as long as the other player keeps playing his/her strategy (cf. definition of Nash equilibria).

Expected payoffs: $(3/2, 3/2)$ instead of $(2/3, 2/3)$. 
Observations and Information Partitions

We assume that observations are made based on a finite probability space $(\Omega, \pi)$, where $\Omega$ is a set of states and $\pi$ is a probability measure on $\Omega$.

Agents might not be able to distinguish all states from each other. In order to model this, we assume for each player $i$ an information partition $P_i = \{P_{i1}, P_{i2}, \ldots, P_{ik}\}$. This means that $\bigcup_{j=1}^{ik} P_j = \Omega$ and for all $P_i, P_k \in P_i$ with $P_j \neq P_k$, we have $P_j \cap P_k = \emptyset$.

**Example:** $\Omega = \{x, y, z\}$, $P_1 = \{\{x\}, \{y, z\}\}$, $P_2 = \{\{x, y\}, \{z\}\}$.

We say that a function $f : \Omega \rightarrow X$ respects an information partition for player $i$ if $f(\omega) = f(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ for some $P_i \in P_i$.

**Example:** $f$ respects $P_1$ if $f(y) = f(z)$.
Definition

A correlated equilibrium of a strategic game \( \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \) consists of

- a finite probability space \((\Omega, \pi)\),
- for each player \(i \in N\) an information partition \(\mathcal{P}_i\) of \(\Omega\),
- for each player \(i \in N\) a function \(\sigma_i : \Omega \rightarrow A_i\) that respects \(\mathcal{P}_i\) (\(\sigma_i\) is player \(i\)'s strategy)

such that for every \(i \in N\) and every function \(\tau_i : \Omega \rightarrow A_i\) that respects \(\mathcal{P}_i\) (i.e. for every possible strategy of player \(i\)) we have

\[
\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)). \quad (2)
\]
Equilibria: \((T, R)\) with \((2, 7)\), \((B, L)\) with \((7, 2)\), and mixed \(((\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}))\) with \((4\frac{2}{3}, 4\frac{2}{3})\).

Assume \(\Omega = \{x, y, z\}\), \(\pi(x) = \frac{1}{3}\), \(\pi(y) = \frac{1}{3}\), \(\pi(z) = \frac{1}{3}\).
Assume further \(P_1 = \{\{x\}, \{y, z\}\}\), \(P_2 = \{\{x, y\}, \{z\}\}\).
Set \(\sigma_1(x) = B, \sigma_1(y) = \sigma_1(z) = T\) and \(\sigma_2(x) = \sigma_2(y) = L, \sigma_2(z) = R\).

Then both player play optimally and get a payoff profile of \((5, 5)\).
Connection to Nash Equilibria

Proposition

For every mixed strategy Nash equilibrium $\alpha$ of a finite strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, there is a correlated equilibrium $\langle (\Omega, \pi), (P_i), (\sigma_i) \rangle$ in which for each player $i$ the distribution on $A_i$ induced by $\sigma_i$ is $\alpha_i$.

This means that correlated equilibria are a generalization of Nash equilibria.
Proof

Let $\Omega = A$ and define $\pi(a) = \prod_{j \in N} \alpha_j(a_j)$. For each player $i$, let $a \in P$ and $b \in P$ for $P \in \mathcal{P}_i$ if $a_i = b_i$. Define $\sigma_i(a) = a_i$ for each $a \in A$.

Then $\langle (\Omega, \pi), (\mathcal{P}_i), (\sigma_i) \rangle$ is a correlated equilibrium since the left hand side of (2) is the Nash equilibrium payoff and for each player $i$ at least as good any other strategy $\tau_i$ respecting the information partition. Further, the distribution induced by $\sigma_i$ is $\alpha_i$. \qed
Proposition

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. Any convex combination of correlated equilibrium payoff profiles of $G$ is a correlated equilibrium payoff profile of $G$.

Proof idea: From given equilibria and weighting factors, create a new one by combining them orthogonally, using the weighting factors.
**Proof**

Let $u^1, \ldots, u^K$ be the payoff profiles and let $(\lambda^1, \ldots, \lambda^K) \in \mathbb{R}^K$ with $\lambda^l \geq 0$ and $\sum_{l=1}^K \lambda^l = 1$. For each $l$ let $\langle (\Omega^l, \pi^l), (\mathcal{P}^l_i), (\sigma^l_i) \rangle$ be a correlated equilibrium generating payoff $u^l$. Wlog. assume all $\Omega^l$’s are disjoint.

Now we define a correlated equilibrium generating the payoff $\sum_{l=1}^K \lambda^l u^l$. Let $\Omega = \bigcup_l \Omega^l$. For any $\omega \in \Omega$ define $\pi(\omega) = \lambda^l \pi^l(\omega)$ where $l$ is such that $\omega \in \Omega^l$. For each $i \in N$ let $\mathcal{P}_i = \bigcup_l \mathcal{P}^l_i$ and set $\sigma_i(\omega) = \sigma^l_i(\omega)$ where $l$ is such that $\omega \in \Omega^l$.

Basically, first throw a dice for which CE to go for, then proceed in this CE.
4 Summary

Mixed Strategies
Nash's Theorem
Correlated Equilibria

Summary
Summary

- **Mixed strategies** allow randomization.
- **Characterization** of mixed-strategy Nash equilibria: players only play best responses with positive probability (support lemma).
- **Nash’s Theorem**: Every finite strategic game has a mixed-strategy Nash equilibrium.
- **Correlated equilibria** can lead to higher payoffs.
- All Nash equilibria are correlated equilibria, but not *vice versa*. 