Strategic Games

Definition (Strategic game)

A **strategic game** is a tuple $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ where

- a nonempty finite set $N$ of players,
- for each player $i \in N$, a nonempty set $A_i$ of actions (or strategies), and
- for each player $i \in N$, a payoff function $u_i : A \rightarrow \mathbb{R}$, where $A = \prod_{i \in N} A_i$.

A strategic game $G$ is called finite if $A$ is finite.

A **strategy profile** is a tuple $a = (a_1, \ldots, a_{|N|}) \in A$.

We can describe finite strategic games using **payoff matrices**.

**Example:** Two-player game where player 1 has actions $T$ and $B$, and player 2 has actions $L$ and $R$, with payoff matrix

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>L</strong></td>
<td>$w_1, w_2$</td>
<td>$x_1, x_2$</td>
</tr>
<tr>
<td><strong>R</strong></td>
<td>$y_1, y_2$</td>
<td>$z_1, z_2$</td>
</tr>
</tbody>
</table>

**Read:** If player 1 plays $T$ and player 2 plays $L$ then player 1 gets payoff $w_1$ and player 2 gets payoff $w_2$, etc.
Prisoner’s Dilemma

Example (Prisoner’s Dilemma (informally))
Two prisoners are interrogated separately, and have the options to either cooperate (C) with their fellow prisoner and stay silent, or defect (D) and accuse the fellow prisoner of the crime.

Possible outcomes:
- Both cooperate: no hard evidence against either of them, only short prison sentences for both.
- One cooperates, the other defects: the defecting prisoner is set free immediately, and the cooperating prisoner gets a very long prison sentence.
- Both confess: both get medium-length prison sentences.

Hawk and Dove

An anti-coordination game:

Example (Hawk and Dove (informally))
In a fight for resources two players can behave either like a dove (D), yielding, or like a hawk (H), attacking.

Possible outcomes:
- Both players behave like doves: both players share the benefit.
- A hawk meets a dove: the hawk wins and gets the bigger part.
- Both players behave like hawks: the benefit gets lost completely because they will fight each other.
Matching Pennies

A strictly competitive game:

**Example (Matching Pennies (informally))**
Two players can choose either heads (H) or tails (T) of a coin.

**Possible outcomes:**
- **Both players make the same choice:** player 1 receives one Euro from player 2.
- **The players make different choices:** player 2 receives one Euro from player 1.

**Strategies**

\[
A_1 = A_2 = \{H, T\}.
\]

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>T</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

Bach or Stravinsky (aka Battle of the Sexes)

A coordination game:

**Example (Bach or Stravinsky (informally))**
Two persons, one of whom prefers Bach whereas the other prefers Stravinsky want to go to a concert together. For both it is more important to go to the same concert than to go to their favorite one. Let B be the action of going to the Bach concert and S the action of going to the Stravinsky concert.

**Possible outcomes:**
- **Both players make the same choice:** the player whose preferred option is chosen gets high payoff, the other player gets medium payoff.
- **The players make different choices:** they both get zero payoff.

**Strategies**

\[
A_1 = A_2 = \{B, S\}.
\]

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>S</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>
Question: What is a “solution” of a strategic game?

Answer:

- A strategy profile where all players play strategies that are rational (i.e., in some sense optimal).
- Note: There are different ways of making the above item precise (different solution concepts).
- A solution concept is a formal rule for predicting how a game will be played.

In the following, we will consider some solution concepts:

- Iterated dominance
- Nash equilibrium
- (Subgame-perfect equilibrium)

Notation: we want to write down strategy profiles where one player’s strategy is removed or replaced.

Let \( a = (a_1, \ldots, a_N) \in A = \prod_{i \in N} A_i \) be a strategy profile.

We write:

- \( A_{-i} := \prod_{i \in N \setminus \{i\}} A_j \),
- \( a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N) \), and
- \( (a_{-i}, a_i') := (a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_N) \).

Example

Let \( A_1 = \{T, B\}, A_2 = \{L, R\}, A_3 = \{X, Y, Z\} \), and \( a := (T, R, Z) \).

Then \( a_{-1} = (R, Z) \), \( a_{-2} = (T, Z) \), and \( a_{-3} = (T, R) \).

Moreover, \( (a_{-2}, L) = (T, L, Z) \).
3 Dominated Strategies

- Strictly Dominated Strategies
- Weakly Dominated Strategies

**Strictly Dominated Strategies**

**Definition (Strictly dominated strategy)**

Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic game.

A strategy $a_i \in A_i$ is called **strictly dominated** in $G$ if there is a strategy $a_i^+ \in A_i$ such that for all strategy profiles $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

We say that $a_i^+$ **strictly dominates** $a_i$.

If $a_i^+ \in A_i$ strictly dominates every other strategy $a_i' \in A_i \setminus \{a_i^+\}$, we call $a_i^+$ **strictly dominant** in $G$.

**Remark:** Playing strictly domained strategies is irrational.

**Strictly Dominated Strategies**

**Question:** What strategy should an agent avoid?

**One answer:**

- **Eliminate all obviously irrational strategies.**
- A strategy is obviously irrational if there is another strategy that is always better, no matter what the other players do.
Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for the prisoner’s dilemma)

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
</tr>
<tr>
<td>player 1</td>
<td>3,3</td>
</tr>
<tr>
<td>D</td>
<td>4,0</td>
</tr>
</tbody>
</table>

- Step 1: eliminate row C (strictly dominated by row D)
- Step 2: eliminate column C (strictly dominated by col. D)
### Strictly Dominated Strategies

**Example (Iterative elim. of strictly dominated strategies)**

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
</tr>
<tr>
<td>T</td>
<td>2,1</td>
</tr>
<tr>
<td>player 1</td>
<td>M</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
</tr>
</tbody>
</table>

**Steps:**
- **Step 1:** Eliminate row **B** (strictly dominated by row **M**)
- **Step 2:** Eliminate column **R** (strictly dominated by col. **L**)
- **Step 3:** Eliminate row **M** (strictly dominated by row **T**)

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### Strictly Dominated Strategies

**Example (Iterative elim. of strictly dominated strategies)**

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
</tr>
<tr>
<td>T</td>
<td>2,1</td>
</tr>
<tr>
<td>player 1</td>
<td>M</td>
</tr>
</tbody>
</table>

**Steps:**
- **Step 1:** Eliminate row **B** (strictly dominated by row **M**)
- **Step 2:** Eliminate column **R** (strictly dominated by col. **L**)
- **Step 3:** Eliminate row **M** (strictly dominated by row **T**)

---

### Preliminaries and Examples

**Solution Concepts and Notation**

- Dominated Strategies
  - Strictly Dominated Strategies
  - Weakly Dominated Strategies
- Nash Equilibria
- Zero-Sum Games
- Summary
Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies)

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Step 1: eliminate row B (strictly dominated by row M)
Step 2: eliminate column R (strictly dominated by col. L)
Step 3: eliminate row M (strictly dominated by row T)

Remark
Strict dominance between actions is rather rare. We should identify more constraints on “solutions”, better solution concepts.

Proposition
The result of iterative elimination of strictly dominated strategies is unique, i.e., independent of the elimination order.

Proof
Homework.
Weakly Dominated Strategies

Definition (Weakly dominated strategy)

Let \( G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \) be a strategic game.

A strategy \( a_i \in A_i \) is called weakly dominated in \( G \) if there is a strategy \( a_i^+ \in A_i \) such that for all profiles \( a_{-i} \in A_{-i} \),

\[
    u_i(a_{-i}, a_i) \leq u_i(a_{-i}, a_i^+) \quad \text{and} \quad u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).
\]

We say that \( a_i^+ \) weakly dominates \( a_i \).

If \( a_i^+ \in A_i \) weakly dominates every other strategy \( a_i' \in A_i \setminus \{a_i^+\} \),
we call \( a_i^+ \) weakly dominant in \( G \).

Example (Iterative elimination of weakly dominated strategies)

<table>
<thead>
<tr>
<th>player 1</th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
</tr>
<tr>
<td>T</td>
<td>2,1</td>
</tr>
<tr>
<td>M</td>
<td>2,1</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Step 1: eliminate row \( B \) (weakly dominated by row \( M \),
\( u_1(M, L) = 2 > 0 = u_1(B, L) \) and \( u_1(M, R) = 1 = u_1(B, R) \))
Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

\[
\begin{array}{ccc}
\text{player 2} & \text{L} & \text{R} \\
T & 2,1 & 0,0 \\
M & 2,1 & 1,1 \\
B & 0,0 & 1,1 \\
\end{array}
\]

- **Step 1:** eliminate row \( B \) (weakly dominated by row \( M \), \( u_1(M, L) = 2 > 0 = u_1(B, L) \) and \( u_1(M, R) = 1 = u_1(B, R) \))
- **Step 2:** eliminate column \( R \) (weakly dominated by col. \( L \))

Here, two solution profiles remain.

Iterative elimination of weakly dominated strategies:
- leads to smaller games,
- can also lead to situations where only a single solution remains,
- but: the result can depend on the elimination order! (see example on next slide)
### Weakly Dominated Strategies

#### Example (Iterative elim. of weakly dominated strategies)

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>M</strong></td>
<td>2,1</td>
<td>1,1</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>0,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

- **Step 1**: eliminate row *T* (weakly dominated by row *M*)
- **Step 2**: eliminate column *L* (weakly dominated by col. *R*)

Different elimination order, different result, even different payoffs (1,1 vs. 2,1)!
**Nash Equilibria**

**Question:** Which strategy profiles are stable?

**Possible answer:**
- Strategy profiles where no player benefits from playing a different strategy
- Equivalently: Strategy profiles where every player’s strategy is a best response to the other players’ strategies

Such strategy profiles are called Nash equilibria, one of the most-used solution concepts in game theory.

**Remark:** In following examples, for non-Nash equilibria, only one possible profitable deviation is shown (even if there are more).

**Definition (Best response)**
Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game, $i \in N$ a player, and $a_{-i} \in A_{-i}$ a strategy profile of the players other than $i$.
Then a strategy $a_i \in A_i$ is a best response of player $i$ to $a_{-i}$ if

$$u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a'_i) \quad \text{for all } a'_i \in A_i.$$

We write $B_i(a_{-i})$ for the set of best responses of player $i$ to $a_{-i}$.

For a strategy profile $a \in A$, we write $B(a) = \prod_{i \in N} B_i(a_{-i})$.

**Remark:** There is an alternative definition of Nash equilibria (which we consider because it gives us a slightly different perspective on Nash equilibria).

**Definition (Best response)**

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game, $i \in N$ a player, and $a_{-i} \in A_{-i}$ a strategy profile of the players other than $i$.
Then a strategy $a_i \in A_i$ is a best response of player $i$ to $a_{-i}$ if

$$u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a'_i) \quad \text{for all } a'_i \in A_i.$$

We write $B_i(a_{-i})$ for the set of best responses of player $i$ to $a_{-i}$.

For a strategy profile $a \in A$, we write $B(a) = \prod_{i \in N} B_i(a_{-i})$.

**Definition (Nash equilibrium, alternative 1)**
A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that for every player $i \in N$, $a'_i \in B_i(a^*_{-i})$.

**Definition (Nash equilibrium, alternative 2)**
A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that $a^*_i \in B(a^*_{-i})$.

**Proposition**
The three definitions of Nash equilibria are equivalent.

**Proof.**
Homework.
### Nash Equilibria

#### Example (Nash Equilibria in the Prisoner’s Dilemma)

<table>
<thead>
<tr>
<th></th>
<th>player 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>3,3</td>
</tr>
<tr>
<td>D</td>
<td>4,0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0,4</td>
</tr>
<tr>
<td>D</td>
<td>1,1</td>
</tr>
</tbody>
</table>

- \((C, C)\): No Nash equilibrium (player 1: \(C \rightarrow D\))
- \((C, D)\): No Nash equilibrium (player 1: \(C \rightarrow D\))
- \((D, C)\): No Nash equilibrium (player 2: \(C \rightarrow D\))
- \((D, D)\): Nash equilibrium!

#### Example (Nash Equilibria in Hawk and Dove)

<table>
<thead>
<tr>
<th></th>
<th>player 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>3,3</td>
</tr>
<tr>
<td>H</td>
<td>4,1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>1,4</td>
</tr>
<tr>
<td>H</td>
<td>0,0</td>
</tr>
</tbody>
</table>

- \((D, D)\): No Nash equilibrium (player 1: \(D \rightarrow H\))
- \((D, H)\): Nash equilibrium!
- \((H, D)\): Nash equilibrium!
- \((H, H)\): No Nash equilibrium (player 1: \(H \rightarrow D\))

#### Example (Nash Equilibria in Matching Pennies)

<table>
<thead>
<tr>
<th></th>
<th>player 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1, 1</td>
</tr>
<tr>
<td>T</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>-1, 1</td>
</tr>
<tr>
<td>T</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

- \((H, H)\): No Nash equilibrium (player 2: \(H \rightarrow T\))
- \((H, T)\): No Nash equilibrium (player 1: \(H \rightarrow T\))
- \((T, H)\): No Nash equilibrium (player 1: \(T \rightarrow H\))
- \((T, T)\): No Nash equilibrium (player 2: \(T \rightarrow H\))

#### Example (Nash Equilibria in Bach or Stravinsky)

<table>
<thead>
<tr>
<th></th>
<th>Stravinsky enthusiast</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>2,1</td>
</tr>
<tr>
<td>S</td>
<td>0,0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Bach enthusiast</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>2,1</td>
</tr>
<tr>
<td>S</td>
<td>0,0</td>
</tr>
</tbody>
</table>

- \((B, B)\): Nash equilibrium!
- \((B, S)\): No Nash equilibrium (player 1: \(B \rightarrow S\))
- \((S, B)\): No Nash equilibrium (player 2: \(S \rightarrow B\))
- \((S, S)\): Nash equilibrium!
Assume three bidders 1, 2, and 3, with valuations and bids.

Example (First-price sealed-bid auction)

- An object has to be assigned to a winning bidder in exchange for a payment.
- For each player (“bidder”) $i = 1, \ldots, n$, let $v_i$ be the private value that bidder $i$ assigns to the object. (We assume that $v_1 > v_2 > \cdots > v_n > 0$.)
- The bidders simultaneously give their bids $b_i \geq 0$, $i = 1, \ldots, n$.
- The object is given to the bidder $i$ with the highest bid $b_i$.
  
  (Ties are broken in favor of bidders with lower index, i.e., if $b_1 = b_j$ are the highest bids, then bidder $i$ will win if $i < j$.)

Observations:

- Bidder 1 wins, pays 90, gets utility $u_1(b) = v_1 - b_1 = 100 - 90 = 10$.
- Bidders 2 and 3 pay nothing, get utility 0.
- (Bidder 2 over-bids.)
- Bidder 1 could still win, but pay less, by bidding $b'_1 = 85$ instead. Then $u_1(b_{-1}, b'_1) = v_1 - b'_1 = 100 - 85 = 15$.

Question: What should the winning bidder have to pay?

One possible answer: The highest bid.

Definition (First-price sealed-bid auction)

- $N = \{1, \ldots, n\}$ with $v_1 > v_2 > \cdots > v_n > 0$,
- $A_i = \mathbb{R}_0^+$ for all $i \in N$,
- Bidder $i \in N$ wins if $b_i$ is maximal among all bids (+ possible tie-breaking by index), and
- $u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - b_i & \text{otherwise} \end{cases}$
  
  where $b = (b_1, \ldots, b_n)$.

Example (Second-price sealed-bid auction)

Assume three bidders 1, 2, and 3, with valuations and bids $v_1 = 100$, $v_2 = 80$, $v_3 = 53$, $b_1 = 90$, $b_2 = 85$, $b_3 = 45$.

Observations:

- Bidder 1 wins, pays 90, gets utility $u_1(b) = v_1 - b_1 = 100 - 90 = 10$.
- Bidders 2 and 3 pay nothing, get utility 0.
- (Bidder 2 over-bids.)
- Bidder 1 could still win, but pay less, by bidding $b'_1 = 85$ instead. Then $u_1(b_{-1}, b'_1) = v_1 - b'_1 = 100 - 85 = 15$.

Question: How to avoid untruthful bidding and incentivize truthful revelation of private valuations?

Different answer to question about payments: Winner pays the second-highest bid.

Definition (Second-price sealed-bid auction)

- $N = \{1, \ldots, n\}$ with $v_1 > v_2 > \cdots > v_n > 0$,
- $A_i = \mathbb{R}_0^+$ for all $i \in N$,
- Bidder $i \in N$ wins if $b_i$ is maximal among all bids (+ possible tie-breaking by index), and
- $u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - \max b_{-i} & \text{otherwise} \end{cases}$
  
  where $b = (b_1, \ldots, b_n)$. 
Example: Sealed-Bid Auctions

Example (Second-price sealed-bid auction)
Assume three bidders 1, 2, and 3, with valuations and bids
\[ v_1 = 100, \quad v_2 = 80, \quad v_3 = 53, \]
\[ b_1 = 90, \quad b_2 = 85, \quad b_3 = 45. \]

Observations:
- Bidder 1 wins, pays 85, gets utility
  \[ u_1(b) = v_1 - b_2 = 100 - 85 = 15. \]
- Bidders 2 and 3 pay nothing, get utility 0.
- Bidder 1 has no incentive to bid strategically and guess the other bidders' private valuations.

Proof (ctd.)
Ad (1) [regardless of what the other bidders do, \( b_i^* \) is always a best response]:

- Case I) bidder \( i \) wins:
  bidder \( i \) pays \( \max b_{-i} \leq v_i \), gets utility
  \[ u_i(b_{-i}, b_i^*) \geq 0. \]

- Case I.a) bidder \( i \) decreases bid:
  this does not help, since he might still win and pay the same as before, or lose and get utility 0.

- Case I.b) bidder \( i \) increases bid:
  bidder \( i \) still wins and pays the same as before.

Example: Sealed-Bid Auctions

Proposition
In a second-price sealed-bid auction, bidding one's own valuation, \( b_i^* = v_i \), is a weakly dominant strategy.

Proof.
We have to show that \( b_i^* \) weakly dominates every other strategy \( b_i \) of player \( i \).

For that, it suffices to show that
- for all \( b_i \in A_i \), we have
  \[ u_i(b_{-i}, b_i^*) \geq u_i(b_{-i}, b_i) \] for all \( b_{-i} \in A_{-i} \), and that
- for all \( b_i \in A_i \), we have
  \[ u_i(b_{-i}, b_i^*) > u_i(b_{-i}, b_i) \] for at least one \( b_{-i} \in A_{-i} \).
Example: Sealed-Bid Auctions

Proof (ctd.)
Ad (2) [for each alternative \( b_i \) to \( b_i^* \), there is an opponent profile \( b_{-i} \) against which \( b_i^* \) is strictly better than \( b_i \)]:

Let \( b_i \) be some strategy other than \( b_i^* \):

- **Case I** \( b_i < b_i^* \):
  - Consider \( b_{-i} \) with \( b_i < \max b_{-i} < b_i^* \).
  - With \( b_i \), bidder \( i \) does not win any more, i.e., we have \( u_i(b_{-i}, b_i^*) > 0 = u_i(b_{-i}, b_i) \).

- **Case II** \( b_i > b_i^* \):
  - Consider \( b_{-i} \) with \( b_i > \max b_{-i} > b_i^* \).
  - With \( b_i \), bidder \( i \) overbids and pays more than the object is worth to him, i.e., we have \( u_i(b_{-i}, b_i^*) = 0 > u_i(b_{-i}, b_i) \).

Iterative Elimination and Nash Equilibria

**Motivation:** We have seen two different solution concepts,
- Surviving iterative elimination of (strictly) dominated strategies and
- Nash equilibria.

**Obvious question:** Is there any relationship between the two?

**Answer:** Yes, Nash equilibria refine the concept of iterative elimination of strictly dominated strategies. We will formalize this on the next slides.

Example: Sealed-Bid Auctions

**Proposition**
Profiles of weakly dominant strategies are Nash equilibria.

**Proof.**

**Proposition**
In a second-price sealed-bid auction, if all bidders bid their true valuations, this is a Nash equilibrium.

**Proof.**

Follows immediately from the previous two propositions.

**Remark:** This is not the only Nash equilibrium in second-price sealed-bid auctions, though.
Proof (ctd.)

“⇒” Let \( a^* \) be a Nash equilibrium of \( G \).

- Nash equilibrium strategies are not eliminated: For players \( j \neq i \), this is clear, because none of their strategies are eliminated.

  For player \( i \), action \( a_j^* \) is a best response to \( a_{-j}^* \), and in particular at least as good a response as \( a_j^* \):

  \[
  u_i(a_{-j}, a_j^*) \geq u_i(a_{-j}, a_j^*).
  \]

  With (1) \( u_i(a_{-j}, a_j^*) > u_i(a_{-j}, a_j') \), we get \( u_i(a_{-j}', a_j^*) > u_i(a_{-j}', a_j) \) and hence \( a_j^* \neq a_j' \).

  Thus, the Nash equilibrium strategy \( a_j^* \) is not eliminated.

Proof (ctd.)

“⇐” (ctd.):

- For player \( i \): Since \( A_i = A_i^j \cup \{ a_i \} \) and \( a_i^* \) is a best response to \( a_{-j}^* \) among the strategies in \( A_i^j \), it suffices to show that \( a_i \) is no better response.

  Because \( a^* \) is a Nash equilibrium in \( G' \) and \( a_j^* \) is a strategy in \( A_i^j \), we have \( u_i(a_{-j}^*, a_j^*) \geq u_i(a_{-j}^*, a_j^*) \).

  Since \( a_j^* \) strictly dominates \( a_i \), we have

  \[
  u_i(a_{-j}^*, a_j^*) > u_i(a_{-j}^*, a_j),
  \]

  and hence

  \[
  u_i(a_{-j}^*, a_j^*) > u_i(a_{-j}^*, a_i).
  \]

  Therefore, \( a_i \) cannot be a better response to \( a_{-j}^* \) than \( a_j^* \).

  Hence, \( a^* \) is also a Nash equilibrium of \( G \).

Corollary

If iterative elimination of strictly dominated strategies results in a unique strategy profile \( a^* \), then \( a^* \) is the unique Nash equilibrium of the original game.

Proof.

Assume that \( a^* \) is the unique remaining strategy profile. By definition, \( a^* \) must be a Nash equilibrium of the remaining game.

We can inductively apply the previous lemma (preservation of Nash equilibria) and see that \( a^* \) (an no other strategy profile) must have been a Nash equilibrium before the last elimination step, and before that step, \( \ldots \), and in the original game.
5 Zero-Sum Games

Playing it Safe (in Two-Player Games)

Motivation: What happens if both players try to “play it safe”?

Question: What does it even mean to “play it safe”?

Answer: Choose a strategy that guarantees the highest worst-case payoff.

Example

Player 1

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>2, 1</td>
<td>2, -20</td>
</tr>
<tr>
<td>M</td>
<td>3, 0</td>
<td>-10, 1</td>
</tr>
<tr>
<td>B</td>
<td>-100, 2</td>
<td>3, 3</td>
</tr>
</tbody>
</table>

Worst-case payoff for player 1:
- if playing T: 2
- if playing M: -10
- if playing B: -100

Worst-case payoff for player 2:
- if playing L: 0
- if playing R: -20
- play T.

However: Unlike (B, R), the profile (T, L) is not a Nash equilibrium.
Playing it Safe (in Two-Player Games)

Example

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
</tr>
<tr>
<td>player 1</td>
<td>2, 1</td>
</tr>
<tr>
<td>M</td>
<td>3, 0</td>
</tr>
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Worst-case payoff for player 1:
- if playing T: 2
- if playing M: −10
- if playing B: −100
→ play T.

Worst-case payoff for player 2:
- if playing L: 0
- if playing R: −20
→ play L.

However: Unlike (B, R), the profile (T, L) is not a Nash equilibrium.

Zero-Sum Games

Definition (Zero-sum game)
A zero-sum game is a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and
$$u_1(a) = -u_2(a)$$
for all $a \in A$.

Example (Matching Pennies as a zero-sum game)

<table>
<thead>
<tr>
<th></th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>H</td>
</tr>
<tr>
<td>player 1</td>
<td>1, −1</td>
</tr>
<tr>
<td>T</td>
<td>−1, 1</td>
</tr>
</tbody>
</table>

Maximimizers

Definition (Maximimimizer)
Let $G = \langle \{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a zero-sum game. An action $x^* \in A_1$ is called maximimimizer for player 1 in $G$ if
$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y)$$
for all $x \in A_1$, and $y^* \in A_2$ is called maximimimizer for player 2 in $G$ if
$$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y)$$
for all $y \in A_2$. Observation: In general, pairs of maximimizers, like (T, L) in the example above, are not the same as Nash equilibria.

Claim: However, in zero-sum games, pairs of maximimizers and Nash equilibria are essentially the same.

(Tiny restriction: This does not hold if the considered game has no Nash equilibrium at all, because unlike Nash equilibria, pairs of maximimizers always exist.)

Reason (intuitively): In zero-sum games, the worst-case assumption that the other player tries to harm you as much as possible is justified, because harming the other is the same as maximizing ones own payoff. Playing it safe is rational.
Maximinimizers

Example (Zero-sum game with three actions each)

player 1

\[\begin{array}{ccc}
T & L & R \\
8 & -8 & 3 & -3 & -6, 6 \\
M & 2 & -2 & -1 & 1 & 3, -3 \\
B & -6 & 6 & 4 & -4 & 8, -8 \\
\end{array}\]

Guaranteed worst-case payoffs:

- \(T: -6, M: -1, B: -6 \Rightarrow \text{maximinimizer } M\)
- \(L: -8, C: -4, R: -8 \Rightarrow \text{maximinimizer } C\)

\(\sim\) pair of maximimizers \((M, C)\) with payoffs \((-1, 1)\)
(not a Nash equilibrium; this game has no Nash equilibrium.)

Maximinimizers

Example (Maximinimization vs. minimaximization)

player 1

\[\begin{array}{ccc}
T & L & R \\
1 & -1 & 2 & -2 \\
B & -2 & 2 & -4 & 4 \\
\end{array}\]

Worst-case payoffs (player 2):

- \(L: -1, R: -2\)
- \(L: +1, R: +2\)
- Maximize: \(-1\)
- Minimize: \(+1\)

Observation: Results identical up to different sign.

Maximinimizers

Lemma

Let \(G = (\{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N})\) be a zero-sum game. Then

\[
\max_y \min_x u_2(x, y) = -\min_y \max_x u_1(x, y). \tag{2}
\]

Proof.

For any real-valued function \(f\), we have

\[
\min_z -f(z) = -\max_z f(z). \tag{3}
\]

Proof (ctd.)

Thus, for all \(y \in A_2\),

\[
-\min_y \max_x u_1(x, y) \overset{(3)}{=} \max_y -\min_x u_1(x, y) \overset{(3)}{=} \max_y \min_x u_2(x, y). \tag{4}
\]
Nash Equilibria in Zero-Sum Games

Now, we are ready to prove our main theorem about zero-sum games and Nash equilibria.

In zero-sum games:
1. Every Nash equilibrium is a pair of maximinimizers.
2. All Nash equilibria have the same payoffs.
3. If there is at least one Nash equilibrium, then every pair of maximinimizers is a Nash equilibrium.

Proof.
1. Let \((x^*, y^*)\) be a Nash equilibrium. Then
   \[ u_2(x^*, y^*) \geq u_2(x, y) \] for all \(y \in A_2\).

   With \(u_1 = -u_2\), this implies
   \[ u_1(x^*, y^*) \leq u_1(x, y) \] for all \(y \in A_2\).

   Thus
   \[ u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y) \leq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \quad (4) \]

   \[ u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \quad (5) \]
Nash Equilibria in Zero-Sum Games

Proof (ctd.)

2 (ctd.)
Inequalities (4) and (5) together imply that
\[ u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \] (6)

Thus, \( x^* \) is a maximinimizer for player 1.

Similarly, we can show that \( y^* \) is a maximinimizer for player 2:
\[ u_2(x^*, y^*) = \max_{y \in A_2} \min_{x \in A_1} u_2(x, y). \] (7)

We only need to put things together:
\[ \max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = u_1(x^*, y^*) =: v^*. \] (8)

In particular, it follows that all Nash equilibria share the same payoff profile.

Proof (ctd.)

3 (ctd.)
Special cases of (10) and (11) for \( x = x^* \) and \( y = y^* \):
\[ u_1(x^*, y^*) \geq v^* \text{ and } u_2(x^*, y^*) \geq -v^*. \]

With \( u_1 = -u_2 \), the latter is equivalent to \( u_1(x^*, y^*) \leq v^* \),
which gives us
\[ u_1(x^*, y^*) = v^*. \] (12)
Plugging (12) into the right-hand side of (10) gives us

\[ u_1(x^*, y) \geq u_1(x^*, y^*) \text{ for all } y \in A_2. \]

With \( u_1 = -u_2 \), this is equivalent to

\[ u_2(x^*, y) \leq u_2(x^*, y^*) \text{ for all } y \in A_2. \]

In other words, \( y^* \) is a best response to \( x^* \).

Similarly, we can plug (12) into the right-hand side of (11) and obtain

\[ u_2(x, y^*) \geq -u_1(x^*, y^*) \text{ for all } x \in A_1. \]

Again using \( u_1 = -u_2 \), this is equivalent to

\[ u_1(x, y^*) \leq u_1(x^*, y^*) \text{ for all } x \in A_1. \]

In words, \( x^* \) is also a best response to \( y^* \).

Hence, \((x^*, y^*)\) is a Nash equilibrium.

\[ \square \]

Corollary

Let \( G = (\{1, 2\}, (A_i)_{i \in \mathbb{N}}, (u_i)_{i \in \mathbb{N}}) \) be a zero-sum game, and let \((x^*_1, y^*_1)\) and \((x^*_2, y^*_2)\) be two Nash equilibria of \( G \).

Then \((x^*_1, y^*_2)\) and \((x^*_2, y^*_1)\) are also Nash equilibria of \( G \).

In other words: Nash equilibria of zero-sum games can be arbitrarily recombined.
Strategic games are one-shot games of finitely many players with given action sets and payoff functions. Players have perfect information.

Solution concepts: survival of iterative elimination of strictly dominated strategies, Nash equilibria.

Relation between solution concepts: Nash equilibria always survive iterative elimination of strictly dominated strategies.

In zero-sum games, one player's gain is the other player's loss. Thus, playing it safe is rational. Relevant concept: maximinimizers.

Relation to Nash equilibria: In zero-sum games, Nash equilibria are pairs of maximinimizers, and, if at least one Nash equilibrium exists, pairs of maximinimizers are also Nash equilibria.