

# Game Theory

## 2. Strategic Games

Albert-Ludwigs-Universität Freiburg



**UNI  
FREIBURG**

Bernhard Nebel and Robert Mattmüller

Summer semester 2018

# 1 Preliminaries and Examples



Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Definition (Strategic game)

A **strategic game** is a tuple  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  where

- a nonempty finite set  $N$  of **players**,
- for each player  $i \in N$ , a nonempty set  $A_i$  of **actions** (or **strategies**), and
- for each player  $i \in N$ , a **payoff function**  $u_i : A \rightarrow \mathbb{R}$ , where  $A = \prod_{i \in N} A_i$ .

A strategic game  $G$  is called finite if  $A$  is finite.

A **strategy profile** is a tuple  $a = (a_1, \dots, a_{|N|}) \in A$ .

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

We can describe finite strategic games using **payoff matrices**.

**Example:** Two-player game where player 1 has actions  $T$  and  $B$ , and player 2 has actions  $L$  and  $R$ , with payoff matrix

		player 2	
		$L$	$R$
player 1	$T$	$w_1, w_2$	$x_1, x_2$
	$B$	$y_1, y_2$	$z_1, z_2$

**Read:** If player 1 plays  $T$  and player 2 plays  $L$  then player 1 gets payoff  $w_1$  and player 2 gets payoff  $w_2$ , etc.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



## Example (Prisoner's Dilemma (informally))

Two prisoners are interrogated separately, and have the options to either cooperate ( $C$ ) with their fellow prisoner and stay silent, or defect ( $D$ ) and accuse the fellow prisoner of the crime.

### Possible outcomes:

- **Both cooperate:** no hard evidence against either of them, only short prison sentences for both.
- **One cooperates, the other defects:** the defecting prisoner is set free immediately, and the cooperating prisoner gets a very long prison sentence.
- **Both confess:** both get medium-length prison sentences.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Prisoner's Dilemma (payoff matrix))

Strategies  $A_1 = A_2 = \{C, D\}$ .

		player 2	
		<i>C</i>	<i>D</i>
player 1	<i>C</i>	3, 3	0, 4
	<i>D</i>	4, 0	1, 1

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

An anti-coordination game:

## Example (Hawk and Dove (informally))

In a fight for resources two players can behave either like a dove ( $D$ ), yielding, or like a hawk ( $H$ ), attacking.

Possible outcomes:

- **Both players behave like doves:** both players share the benefit.
- **A hawk meets a dove:** the hawk wins and gets the bigger part.
- **Both players behave like hawks:** the benefit gets lost completely because they will fight each other.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Hawk and Dove (payoff matrix))

Strategies  $A_1 = A_2 = \{D, H\}$ .

		player 2	
		<i>D</i>	<i>H</i>
player 1	<i>D</i>	3, 3	1, 4
	<i>H</i>	4, 1	0, 0

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



A strictly competitive game:

## Example (Matching Pennies (informally))

Two players can choose either heads ( $H$ ) or tails ( $T$ ) of a coin.

Possible outcomes:

- Both players make the same choice: player 1 receives one Euro from player 2.
- The players make different choices: player 2 receives one Euro from player 1.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Matching Pennies (payoff matrix))

Strategies  $A_1 = A_2 = \{H, T\}$ .

		player 2	
		<i>H</i>	<i>T</i>
player 1	<i>H</i>	1, -1	-1, 1
	<i>T</i>	-1, 1	1, -1

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

# Bach or Stravinsky (aka Battle of the Sexes)



A coordination game:

## Example (Bach or Stravinsky (informally))

Two persons, one of whom prefers Bach whereas the other prefers Stravinsky want to go to a concert together. For both it is more important to go to the same concert than to go to their favorite one. Let  $B$  be the action of going to the Bach concert and  $S$  the action of going to the Stravinsky concert.

Possible outcomes:

- **Both players make the same choice:** the player whose preferred option is chosen gets high payoff, the other player gets medium payoff.
- **The players make different choices:** they both get zero payoff.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

# Bach or Stravinsky (aka Battle of the Sexes)



## Example (Bach or Stravinsky (payoff matrix))

Strategies  $A_1 = A_2 = \{B, S\}$ .

		Stravinsky enthusiast	
		<i>B</i>	<i>S</i>
Bach enthusiast	<i>B</i>	2, 1	0, 0
	<i>S</i>	0, 0	1, 2

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

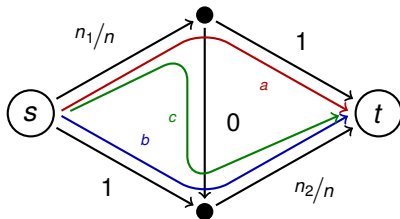
Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (A congestion game)



player 2

		<i>a</i>	<i>b</i>	<i>c</i>
player 1	<i>a</i>	-2, -2	-1.5, -1.5	-2, -1.5
	<i>b</i>	-1.5, -1.5	-2, -2	-2, -1.5
	<i>c</i>	-1.5, -2	-1.5, -2	<b>-2, -2</b>

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

# 2 Solution Concepts and Notation



Preliminaries  
and  
Examples

**Solution  
Concepts  
and Notation**

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

**Question:** What is a “solution” of a strategic game?

**Answer:**

- A strategy profile where all players play strategies that are **rational** (i. e., in some sense optimal).
- **Note:** There are different ways of making the above item precise (different solution concepts).
- A **solution concept** is a formal rule for predicting how a game will be played.

In the following, we will consider some solution concepts:

- Iterated dominance
- Nash equilibrium
- (Subgame-perfect equilibrium)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



**Notation:** we want to write down strategy profiles where one player's strategy is removed or replaced.

Let  $a = (a_1, \dots, a_{|N|}) \in A = \prod_{i \in N} A_i$  be a strategy profile.

**We write:**

- $A_{-i} := \prod_{j \in N \setminus \{i\}} A_j$ ,
- $a_{-i} := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{|N|})$ , and
- $(a_{-i}, a'_i) := (a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_{|N|})$ .

## Example

Let  $A_1 = \{T, B\}$ ,  $A_2 = \{L, R\}$ ,  $A_3 = \{X, Y, Z\}$ , and  $a := (T, R, Z)$ .

Then  $a_{-1} = (R, Z)$ ,  $a_{-2} = (T, Z)$ ,  $a_{-3} = (T, R)$ .

Moreover,  $(a_{-2}, L) = (T, L, Z)$ .



# 3 Dominated Strategies



- Strictly Dominated Strategies
- Weakly Dominated Strategies

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

**Dominated  
Strategies**

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

**Question:** What strategy should an agent avoid?

**One answer:**

- **Eliminate** all obviously **irrational strategies**.
- A strategy is obviously **irrational** if there is **another strategy that is always better**, no matter what the other players do.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Definition (Strictly dominated strategy)

Let  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a strategic game.

A strategy  $a_i \in A_i$  is called **strictly dominated** in  $G$  if there is a strategy  $a_i^+ \in A_i$  such that for all strategy profiles  $a_{-i} \in A_{-i}$ ,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

We say that  $a_i^+$  **strictly dominates**  $a_i$ .

If  $a_i^+ \in A_i$  strictly dominates every other strategy  $a_i' \in A_i \setminus \{a_i^+\}$ , we call  $a_i^+$  **strictly dominant** in  $G$ .

**Remark:** Playing strictly dominated strategies is irrational.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies  
Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

This suggests a solution concept:

**iterative elimination of strictly dominated strategies:**

**while** some strictly dominated strategy is left:

eliminate some strictly dominated strategy

**if** a unique strategy profile remains:

this unique profile is the solution

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

		player 2	
		<i>C</i>	<i>D</i>
player 1	<i>C</i>	3, 3	0, 4
	<i>D</i>	4, 0	1, 1

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

**Strictly Dominated  
Strategies**

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

		player 2	
		<i>C</i>	<i>D</i>
player 1	<del><i>C</i></del>	<del>3, 3</del>	<del>0, 4</del>
	<i>D</i>	4, 0	1, 1

- **Step 1:** eliminate row *C* (strictly dominated by row *D*)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

		player 2	
		<del>C</del>	D
player 1	<del>C</del>	<del>3, 3</del>	<del>0, 4</del>
	D	<del>4, 0</del>	1, 1

- **Step 1:** eliminate row *C* (strictly dominated by row *D*)
- **Step 2:** eliminate column *C* (strictly dominated by col. *D*)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

		player 2	
		<del>C</del>	D
player 1	<del>C</del>	<del>3, 3</del>	<del>0, 4</del>
	D	<del>4, 0</del>	1, 1

- Step 1: eliminate row *C* (strictly dominated by row *D*)
- Step 2: eliminate column *C* (strictly dominated by col. *D*)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary





## Example (Iterative elim. of strictly dominated strategies)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2,1	0,0
	<i>M</i>	1,2	2,1
	<i>B</i>	0,0	1,1

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

**Strictly Dominated  
Strategies**

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Iterative elim. of strictly dominated strategies)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	0, 0
	<i>M</i>	1, 2	2, 1
	<del><i>B</i></del>	<del>0, 0</del>	<del>1, 1</del>

- **Step 1:** eliminate row *B* (strictly dominated by row *M*)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Iterative elim. of strictly dominated strategies)

		player 2	
		L	<del>R</del>
player 1	T	2, 1	<del>0, 0</del>
	M	1, 2	<del>2, 1</del>
	<del>B</del>	<del>0, 0</del>	<del>1, 1</del>

- **Step 1:** eliminate row *B* (strictly dominated by row *M*)
- **Step 2:** eliminate column *R* (strictly dominated by col. *L*)

## Example (Iterative elim. of strictly dominated strategies)

		player 2	
		L	<del>R</del>
player 1	T	2, 1	<del>0, 0</del>
	<del>M</del>	<del>1, 2</del>	<del>2, 1</del>
	<del>B</del>	<del>0, 0</del>	<del>1, 1</del>

- Step 1: eliminate row *B* (strictly dominated by row *M*)
- Step 2: eliminate column *R* (strictly dominated by col. *L*)
- Step 3: eliminate row *M* (strictly dominated by row *T*)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Iterative elim. of strictly dominated strategies)

		player 2	
		L	<del>R</del>
player 1	T	2, 1	<del>0, 0</del>
	<del>M</del>	<del>1, 2</del>	<del>2, 1</del>
	<del>B</del>	<del>0, 0</del>	<del>1, 1</del>

- Step 1: eliminate row *B* (strictly dominated by row *M*)
- Step 2: eliminate column *R* (strictly dominated by col. *L*)
- Step 3: eliminate row *M* (strictly dominated by row *T*)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

Example (Iterative elimination of strictly dominated strategies for Bach or Stravinsky)

		Stravinsky enthusiast	
		<i>B</i>	<i>S</i>
Bach enthusiast	<i>B</i>	2, 1	0, 0
	<i>S</i>	0, 0	1, 2

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

Example (Iterative elimination of strictly dominated strategies for Bach or Stravinsky)

		Stravinsky enthusiast	
		<i>B</i>	<i>S</i>
Bach enthusiast	<i>B</i>	2, 1	0, 0
	<i>S</i>	0, 0	1, 2

- No strictly dominated strategies.
- All strategies survive iterative elimination of strictly dominated strategies.
- All strategies **rationalizable**.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Remark

Strict dominance between actions is rather rare.

We should identify more constraints on “solutions”, better solution concepts.

## Proposition

The result of iterative elimination of strictly dominated strategies is unique, i. e., independent of the elimination order.

## Proof.

Homework. □

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

**Strictly Dominated  
Strategies**

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



## Definition (Weakly dominated strategy)

Let  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a strategic game.

A strategy  $a_i \in A_i$  is called **weakly dominated** in  $G$  if there is a strategy  $a_i^+ \in A_i$  such that for all profiles  $a_{-i} \in A_{-i}$ ,

$$u_i(a_{-i}, a_i) \leq u_i(a_{-i}, a_i^+)$$

and that for at least one profile  $a_{-i} \in A_{-i}$ ,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

We say that  $a_i^+$  **weakly dominates**  $a_i$ .

If  $a_i^+ \in A_i$  weakly dominates every other strategy  $a_i' \in A_i \setminus \{a_i^+\}$ , we call  $a_i^+$  **weakly dominant** in  $G$ .

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

What about  
iterative elimination of weakly dominated strategies  
as a solution concept?  
Let's see what happens.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



## Example (Iterative elim. of weakly dominated strategies)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	0, 0
	<i>M</i>	2, 1	1, 1
	<i>B</i>	0, 0	1, 1

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Iterative elim. of weakly dominated strategies)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	0, 0
	<i>M</i>	2, 1	1, 1
	<del><i>B</i></del>	<del>0, 0</del>	<del>1, 1</del>

- **Step 1:** eliminate row *B* (weakly dominated by row *M*,  $u_1(M, L) = 2 > 0 = u_1(B, L)$  and  $u_1(M, R) = 1 = u_1(B, R)$ )

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	<del>R</del>
player 1	T	2, 1	<del>0, 0</del>
	M	2, 1	<del>1, 1</del>
	<del>B</del>	<del>0, 0</del>	<del>1, 1</del>

- **Step 1:** eliminate row *B* (weakly dominated by row *M*,  $u_1(M, L) = 2 > 0 = u_1(B, L)$  and  $u_1(M, R) = 1 = u_1(B, R)$ )
- **Step 2:** eliminate column *R* (weakly dominated by col. *L*)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	<del>R</del>
player 1	T	2, 1	<del>0, 0</del>
	M	2, 1	<del>1, 1</del>
	<del>B</del>	<del>0, 0</del>	<del>1, 1</del>

- **Step 1:** eliminate row *B* (weakly dominated by row *M*,  $u_1(M, L) = 2 > 0 = u_1(B, L)$  and  $u_1(M, R) = 1 = u_1(B, R)$ )
- **Step 2:** eliminate column *R* (weakly dominated by col. *L*)

Here, two solution profiles remain.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Iterative elimination of weakly dominated strategies:

- leads to **smaller games**,
- can also lead to situations where only a single solution remains,
- **but**: the result can depend on the elimination order!  
(see example on next slide)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



## Example (Iterative elim. of weakly dominated strategies)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	0, 0
	<i>M</i>	2, 1	1, 1
	<i>B</i>	0, 0	1, 1

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

**Weakly Dominated  
Strategies**

Nash  
Equilibria

Zero-Sum  
Games

Summary



## Example (Iterative elim. of weakly dominated strategies)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<del><i>T</i></del>	<del>2, 1</del>	<del>0, 0</del>
	<i>M</i>	2, 1	1, 1
	<i>B</i>	0, 0	1, 1

- **Step 1:** eliminate row *T* (weakly dominated by row *M*)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Iterative elim. of weakly dominated strategies)

		player 2	
		<del>L</del>	R
player 1	<del>T</del>	<del>2, 1</del>	<del>0, 0</del>
	M	<del>2, 1</del>	1, 1
	B	<del>0, 0</del>	1, 1

- Step 1: eliminate row *T* (weakly dominated by row *M*)
- Step 2: eliminate column *L* (weakly dominated by col. *R*)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Iterative elim. of weakly dominated strategies)

		player 2	
		<del>L</del>	R
player 1	<del>T</del>	<del>2, 1</del>	<del>0, 0</del>
	M	<del>2, 1</del>	1, 1
	B	<del>0, 0</del>	1, 1

- Step 1: eliminate row  $T$  (weakly dominated by row  $M$ )
- Step 2: eliminate column  $L$  (weakly dominated by col.  $R$ )

Different elimination order, different result,  
even different payoffs (1, 1 vs. 2, 1)!

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Strictly Dominated  
Strategies

Weakly Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

# 4 Nash Equilibria

- Definitions and Examples
- Example: Sealed-Bid Auctions
- Iterative Elimination and Nash Equilibria

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

**Nash  
Equilibria**

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

Question: Which strategy profiles are **stable**?

Possible answer:

- Strategy profiles where **no player benefits from playing a different strategy**
- **Equivalently**: Strategy profiles where every player's strategy is a **best response** to the other players' strategies

Such strategy profiles are called **Nash equilibria**, one of the **most-used solution concepts** in game theory.

**Remark**: In following examples, for non-Nash equilibria, only one possible profitable deviation is shown (even if there are more).

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

## Definition (Nash equilibrium)

A **Nash equilibrium** of a strategic game  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  is a strategy profile  $a^* \in A$  such that for every player  $i \in N$ ,

$$u_i(a^*) \geq u_i(a_{-i}^*, a_j) \quad \text{for all } a_j \in A_j.$$

**Remark:** There is an alternative definition of Nash equilibria (which we consider because it gives us a slightly different perspective on Nash equilibria).

## Definition (Best response)

Let  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a strategic game,  $i \in N$  a player, and  $a_{-i} \in A_{-i}$  a strategy profile of the players other than  $i$ .

Then a strategy  $a_i \in A_i$  is a **best response** of player  $i$  to  $a_{-i}$  if

$$u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a'_i) \quad \text{for all } a'_i \in A_i.$$

We write  $B_i(a_{-i})$  for the set of best responses of player  $i$  to  $a_{-i}$ .

For a strategy profile  $a \in A$ , we write  $B(a) = \prod_{i \in N} B_i(a_{-i})$ .

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

## Definition (Nash equilibrium, alternative 1)

A **Nash equilibrium** of a strategic game  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  is a strategy profile  $a^* \in A$  such that for every player  $i \in N$ ,  $a_i^* \in B_i(a_{-i}^*)$ .

## Definition (Nash equilibrium, alternative 2)

A **Nash equilibrium** of a strategic game  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  is a strategy profile  $a^* \in A$  such that  $a^* \in B(a^*)$ .

## Proposition

The three definitions of Nash equilibria are equivalent.

## Proof.

Homework. □

Preliminaries and Examples

Solution Concepts and Notation

Dominated Strategies

Nash Equilibria

Definitions and Examples

Example: Sealed-Bid Auctions

Iterative Elimination and Nash Equilibria

Zero-Sum Games

Summary



## Example (Nash Equilibria in the Prisoner's Dilemma)

		player 2	
		<i>C</i>	<i>D</i>
player 1	<i>C</i>	3, 3	0, 4
	<i>D</i>	4, 0	1, 1

- $(C, C)$ : No Nash equilibrium (player 1:  $C \rightarrow D$ )
- $(C, D)$ : No Nash equilibrium (player 1:  $C \rightarrow D$ )
- $(D, C)$ : No Nash equilibrium (player 2:  $C \rightarrow D$ )
- $(D, D)$ : Nash equilibrium!

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

## Example (Nash Equilibria in Hawk and Dove)

		player 2	
		<i>D</i>	<i>H</i>
player 1	<i>D</i>	3, 3	1, 4
	<i>H</i>	4, 1	0, 0

- $(D, D)$ : No Nash equilibrium (player 1:  $D \rightarrow H$ )
- $(D, H)$ : Nash equilibrium!
- $(H, D)$ : Nash equilibrium!
- $(H, H)$ : No Nash equilibrium (player 1:  $H \rightarrow D$ )

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

## Example (Nash Equilibria in Matching Pennies)

		player 2	
		<i>H</i>	<i>T</i>
player 1	<i>H</i>	1, -1	-1, 1
	<i>T</i>	-1, 1	1, -1

- $(H, H)$ : No Nash equilibrium (player 2:  $H \rightarrow T$ )
- $(H, T)$ : No Nash equilibrium (player 1:  $H \rightarrow T$ )
- $(T, H)$ : No Nash equilibrium (player 1:  $T \rightarrow H$ )
- $(T, T)$ : No Nash equilibrium (player 2:  $T \rightarrow H$ )

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

## Example (Nash Equilibria in Bach or Stravinsky)

		Stravinsky enthusiast	
		<i>B</i>	<i>S</i>
Bach enthusiast	<i>B</i>	2,1	0,0
	<i>S</i>	0,0	1,2

- $(B, B)$ : Nash equilibrium!
- $(B, S)$ : No Nash equilibrium (player 1:  $B \rightarrow S$ )
- $(S, B)$ : No Nash equilibrium (player 2:  $S \rightarrow B$ )
- $(S, S)$ : Nash equilibrium!

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

# Example: Sealed-Bid Auctions



We consider a slightly larger example: **sealed-bid auctions**

Setting:

- An **object** has to be **assigned** to a winning bidder in exchange for a **payment**.
- For each player (“bidder”)  $i = 1, \dots, n$ , let  $v_i$  be the **private value** that bidder  $i$  assigns to the object.  
(We assume that  $v_1 > v_2 > \dots > v_n > 0$ .)
- The bidders simultaneously give their **bids**  $b_i \geq 0$ ,  $i = 1, \dots, n$ .
- The object is given to the bidder  $i$  with the **highest bid**  $b_i$ .  
(Ties are broken in favor of bidders with lower index, i.e., if  $b_i = b_j$  are the highest bids, then bidder  $i$  will win iff  $i < j$ .)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

# Example: Sealed-Bid Auctions



**Question:** What should the winning bidder have to **pay**?

**One possible answer:** The highest bid.

## Definition (First-price sealed-bid auction)

- $N = \{1, \dots, n\}$  with  $v_1 > v_2 > \dots > v_n > 0$ ,
- $A_i = \mathbb{R}_0^+$  for all  $i \in N$ ,
- Bidder  $i \in N$  **wins** if  $b_i$  is maximal among all bids (+ possible tie-breaking by index), and
- $$u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - b_i & \text{otherwise} \end{cases}$$
 where  $b = (b_1, \dots, b_n)$ .

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

# Example: Sealed-Bid Auctions



## Example (First-price sealed-bid auction)

Assume three bidders 1, 2, and 3, with valuations and bids

$$\begin{array}{lll} v_1 = 100, & v_2 = 80, & v_3 = 53, \\ b_1 = 90, & b_2 = 85, & b_3 = 45. \end{array}$$

### Observations:

- Bidder 1 wins, pays 90, gets utility  $u_1(b) = v_1 - b_1 = 100 - 90 = 10$ .
- Bidders 2 and 3 pay nothing, get utility 0.
- (Bidder 2 over-bids.)
- Bidder 1 could still win, but pay less, by bidding  $b'_1 = 85$  instead. Then  $u_1(b_{-1}, b'_1) = v_1 - b'_1 = 100 - 85 = 15$ .

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

# Example: Sealed-Bid Auctions



Question: How to avoid **untruthful bidding** and **incentivize truthful revelation** of private valuations?

Different answer to question about payments: Winner pays the **second-highest** bid.

## Definition (Second-price sealed-bid auction)

- $N = \{1, \dots, n\}$  with  $v_1 > v_2 > \dots > v_n > 0$ ,
- $A_i = \mathbb{R}_0^+$  for all  $i \in N$ ,
- Bidder  $i \in N$  **wins** if  $b_i$  is maximal among all bids (+ possible tie-breaking by index), and
- $$u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - \max_{j \neq i} b_j & \text{otherwise} \end{cases}$$
 where  $b = (b_1, \dots, b_n)$ .

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary



## Example (Second-price sealed-bid auction)

Assume three bidders 1, 2, and 3, with valuations and bids

$$\begin{array}{lll} v_1 = 100, & v_2 = 80, & v_3 = 53, \\ b_1 = 90, & b_2 = 85, & b_3 = 45. \end{array}$$

### Observations:

- Bidder 1 wins, pays 85, gets utility  $u_1(b) = v_1 - b_2 = 100 - 85 = 15$ .
- Bidders 2 and 3 pay nothing, get utility 0.
- Bidder 1 has no incentive to bid strategically and guess the other bidders' private valuations.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

# Example: Sealed-Bid Auctions



## Proposition

In a second-price sealed-bid auction, bidding one's own valuation,  $b_i^+ = v_i$ , is a weakly dominant strategy.

## Proof.

We have to show that  $b_i^+$  weakly dominates **every** other strategy  $b_i$  of player  $i$ .

For that, it suffices to show that

1 for all  $b_{-i} \in A_{-i}$ , we have

$$u_i(b_{-i}, b_i^+) \geq u_i(b_{-i}, b_i) \text{ for all } b_{-i} \in A_{-i}, \text{ and that}$$

2 for all  $b_{-i} \in A_{-i}$ , we have

$$u_i(b_{-i}, b_i^+) > u_i(b_{-i}, b_i) \text{ for at least one } b_{-i} \in A_{-i}.$$

Preliminaries and Examples

Solution Concepts and Notation

Dominated Strategies

Nash Equilibria

Definitions and Examples

Example: Sealed-Bid Auctions

Iterative Elimination and Nash Equilibria

Zero-Sum Games

Summary

## Proof (ctd.)

Ad (1) [regardless of what the other bidders do,  
 $b_i^+$  is always a best response]:

- Case I) bidder  $i$  wins:

bidder  $i$  pays  $\max b_{-i} \leq v_i$ , gets  $u_i(b_{-i}, b_i^+) \geq 0$ .

- Case I.a) bidder  $i$  decreases bid:

this does not help, since he might still win and pay the same as before, or lose and get utility 0.

- Case I.b) bidder  $i$  increases bid:

bidder  $i$  still wins and pays the same as before.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

## Proof (ctd.)

### Ad (1) (ctd.):

- **Case II) bidder  $i$  loses:**

bidder  $i$  pays nothing, gets  $u_i(b_{-i}, b_i^+) = 0$ .

- **Case II.a) bidder  $i$  decreases bid:**

bidder  $i$  still loses and gets utility 0.

- **Case II.b) bidder  $i$  increases bid:**

either bidder  $i$  still loses and gets utility 0, or becomes the winner and pays more than the object is worth to him, leading to a negative utility.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

# Example: Sealed-Bid Auctions

## Proof (ctd.)

Ad (2) [for each alternative  $b_i$  to  $b_i^+$ , there is an opponent profile  $b_{-i}$  against which  $b_i^+$  is strictly better than  $b_i$ ]:

Let  $b_i$  be some strategy other than  $b_i^+$ .

■ **Case I)  $b_i < b_i^+$ :**

Consider  $b_{-i}$  with  $b_i < \max b_{-i} < b_i^+$ .

With  $b_i$ , bidder  $i$  does not win any more, i. e., we have

$$u_i(b_{-i}, b_i^+) > 0 = u_i(b_{-i}, b_i).$$

■ **Case II)  $b_i > b_i^+$ :**

Consider  $b_{-i}$  with  $b_i > \max b_{-i} > b_i^+$ .

With  $b_i$ , bidder  $i$  overbids and pays more than the object is

worth to him, i. e., we have  $u_i(b_{-i}, b_i^+) = 0 > u_i(b_{-i}, b_i)$ .

Preliminaries and Examples

Solution Concepts and Notation

Dominated Strategies

Nash Equilibria

Definitions and Examples

Example: Sealed-Bid Auctions

Iterative Elimination and Nash Equilibria

Zero-Sum Games

Summary



# Example: Sealed-Bid Auctions



## Proposition

Profiles of weakly dominant strategies are Nash equilibria.

## Proof.

Homework. □

## Proposition

In a second-price sealed-bid auction, if all bidders bid their true valuations, this is a Nash equilibrium.

## Proof.

Follows immediately from the previous two propositions. □

**Remark:** This is not the only Nash equilibrium in second-price sealed-bid auctions, though.

Preliminaries and Examples

Solution Concepts and Notation

Dominated Strategies

Nash Equilibria

Definitions and Examples

Example: Sealed-Bid Auctions

Iterative Elimination and Nash Equilibria

Zero-Sum Games

Summary



**Motivation:** We have seen **two different solution concepts**,

- Surviving iterative elimination of (strictly) **dominated strategies** and
- **Nash equilibria**.

**Obvious question:** Is there any **relationship** between the two?

**Answer:** Yes, Nash equilibria refine the concept of iterative elimination of strictly dominated strategies. We will formalize this on the next slides.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary



## Lemma (preservation of Nash equilibria)

*Let  $G$  and  $G'$  be two strategic games where  $G'$  is obtained from  $G$  by elimination of one strictly dominated strategy.*

*Then a strategy profile  $a^*$  is a Nash equilibrium of  $G$  if and only if it is Nash equilibrium of  $G'$ .*

## Proof.

Let  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  and  $G' = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$ .

Let  $a'_j$  be the eliminated strategy.

Then there is a strategy  $a_j^+$  such that for all  $a_{-j} \in A_{-j}$ ,

$$u_j(a_{-j}, a'_j) < u_j(a_{-j}, a_j^+). \quad (1)$$

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples  
Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary





## Proof (ctd.)

“ $\Rightarrow$ ”: Let  $a^*$  be a Nash equilibrium of  $G$ .

- **Nash equilibrium strategies are not eliminated:** For players  $j \neq i$ , this is clear, because none of their strategies are eliminated.

For player  $i$ , action  $a_i^*$  is a best response to  $a_{-i}^*$ , and in particular at least as good a response as  $a_i^+$ :

$$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+).$$

With (1)  $u_i(a_{-i}, a_i^+) > u_i(a_{-i}, a_i')$ , we get  $u_i(a_{-i}^*, a_i^+) > u_i(a_{-i}^*, a_i')$  and hence  $a_i^* \neq a_i'$ .

Thus, the Nash equilibrium strategy  $a_i^*$  is not eliminated.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions  
and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary



## Proof (ctd.)

“ $\Rightarrow$ ” (ctd.):

- **Best responses remain best responses:** For all players  $j \in N$ ,  $a_j^*$  is a best response to  $a_{-j}^*$  in  $G$ . Since in  $G'$ , no potentially better responses are introduced ( $A'_j \subseteq A_j$ ) and the payoffs are unchanged, this also holds in  $G'$ .

Hence,  $a^*$  is also a Nash equilibrium of  $G'$ .

“ $\Leftarrow$ ”: Let  $a^*$  be a Nash equilibrium of  $G'$ .

- **For player  $j \neq i$ :**  $a_j^*$  is a best response to  $a_{-j}^*$  in  $G$  as well, since the responses available to player  $j$  in  $G$  and  $G'$  are the same.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary



## Proof (ctd.)

“ $\Leftarrow$ ” (ctd.):

- For player  $i$ : Since  $A_i = A'_i \cup \{a_i\}$  and  $a_i^*$  is a best response to  $a_{-i}^*$  among the strategies in  $A'_i$ , it suffices to show that  $a_i$  is no better response.

Because  $a^*$  is a Nash equilibrium in  $G'$  and  $a_i^+$  is a strategy in  $A'_i$ , we have  $u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+)$ .

Since  $a_i^+$  strictly dominates  $a_i$ , we have  $u_i(a_{-i}^*, a_i^+) > u_i(a_{-i}^*, a_i)$ , and hence  $u_i(a_{-i}^*, a_i^*) > u_i(a_{-i}^*, a_i)$ .

Therefore,  $a_i$  cannot be a better response to  $a_{-i}^*$  than  $a_i^*$ .

Hence,  $a^*$  is also a Nash equilibrium of  $G$ . □

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary



## Corollary

If iterative elimination of strictly dominated strategies results in a *unique* strategy profile  $a^*$ , then  $a^*$  is the unique Nash equilibrium of the original game.

## Proof.

Assume that  $a^*$  is the unique remaining strategy profile. By definition,  $a^*$  must be a Nash equilibrium of the remaining game.

We can inductively apply the previous lemma (preservation of Nash equilibria) and see that  $a^*$  (and no other strategy profile) must have been a Nash equilibrium before the last elimination step, and before that step,  $\dots$ , and in the original game.  $\square$

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Definitions and  
Examples

Example:  
Sealed-Bid  
Auctions

Iterative  
Elimination and  
Nash Equilibria

Zero-Sum  
Games

Summary

# 5 Zero-Sum Games



Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

**Zero-Sum  
Games**

Summary

# Playing it Safe (in Two-Player Games)



**Motivation:** What happens if both players try to “play it safe”?

**Question:** What does it even mean to “play it safe”?

**Answer:** Choose a strategy that guarantees the **highest worst-case payoff**.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

# Playing it Safe (in Two-Player Games)

## Example

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	2, -20
	<i>M</i>	3, 0	-10, 1
	<i>B</i>	-100, 2	3, 3

Worst-case payoff for **player 1**:

- if playing *T*: 2
- if playing *M*: -10
- if playing *B*: -100

↪ play *T*.

Worst-case payoff for **player 2**:

- if playing *L*: 0
- if playing *R*: -20

↪ play *L*.

**However:** Unlike  $(B, R)$ , the profile  $(T, L)$  is **not** a Nash equilibrium.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

# Playing it Safe (in Two-Player Games)

## Example

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	2, -20
	<i>M</i>	3, 0	-10, 1
	<i>B</i>	-100, 2	3, 3

Worst-case payoff for player 1:

- if playing *T*: 2
- if playing *M*: -10
- if playing *B*: -100

↪ play *T*.

Worst-case payoff for player 2:

- if playing *L*: 0
- if playing *R*: -20

↪ play *L*.

**However:** Unlike  $(B, R)$ , the profile  $(T, L)$  is **not** a Nash equilibrium.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



# Playing it Safe (in Two-Player Games)

## Example

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	2, -20
	<i>M</i>	3, 0	-10, 1
	<i>B</i>	-100, 2	3, 3

Worst-case payoff for player 1:

- if playing *T*: 2
- if playing *M*: -10
- if playing *B*: -100

↪ play *T*.

Worst-case payoff for player 2:

- if playing *L*: 0
- if playing *R*: -20

↪ play *L*.

**However:** Unlike  $(B, R)$ , the profile  $(T, L)$  is **not** a Nash equilibrium.

# Playing it Safe (in Two-Player Games)



**Observation:** In general, pairs of **maximinimizers**, like  $(T, L)$  in the example above, are **not** the same as Nash equilibria.

**Claim:** However, in **zero-sum games**, pairs of maximinimizers and Nash equilibria **are essentially the same**.

(Tiny restriction: This does not hold if the considered game has no Nash equilibrium at all, because unlike Nash equilibria, pairs of maximinimizers always exist.)

**Reason (intuitively):** In **zero-sum games**, the **worst-case assumption** that the other player tries to harm you as much as possible is **justified**, because harming the other is the same as maximizing ones own payoff. **Playing it safe is rational.**

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Definition (Zero-sum game)

A **zero-sum game** is a strategic game  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  with  $N = \{1, 2\}$  and

$$u_1(a) = -u_2(a)$$

for all  $a \in A$ .

## Example (Matching Pennies as a zero-sum game)

		player 2	
		<i>H</i>	<i>T</i>
player 1	<i>H</i>	1, -1	-1, 1
	<i>T</i>	-1, 1	1, -1

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Definition (Maximinimizer)

Let  $G = \langle \{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a zero-sum game.

An action  $x^* \in A_1$  is called **maximinimizer** for player 1 in  $G$  if

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y) \quad \text{for all } x \in A_1,$$

and  $y^* \in A_2$  is called **maximinimizer** for player 2 in  $G$  if

$$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y) \quad \text{for all } y \in A_2.$$

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Zero-sum game with three actions each)

		player 2		
		<i>L</i>	<i>C</i>	<i>R</i>
player 1	<i>T</i>	8, -8	3, -3	-6, 6
	<i>M</i>	2, -2	-1, 1	3, -3
	<i>B</i>	-6, 6	4, -4	8, -8

### Guaranteed worst-case payoffs:

- $T: -6, M: -1, B: -6 \rightsquigarrow$  maximinimizer  $M$
- $L: -8, C: -4, R: -8 \rightsquigarrow$  maximinimizer  $C$

$\rightsquigarrow$  pair of maximinimizers ( $M, C$ ) with payoffs  $(-1, 1)$   
(not a Nash equilibrium; this game has no Nash equilibrium.)

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Example (Maximinimization vs. minimaximization)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	1, -1	2, -2
	<i>B</i>	-2, 2	-4, 4

Worst-case payoffs (player 2):

- $L: -1, R: -2$
- Maximize:  $-1$

Best-case payoffs (player 1):

- $L: +1, R: +2$
- Minimize:  $+1$

**Observation:** Results identical up to different sign.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Lemma

Let  $G = \langle \{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a zero-sum game. Then

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) = - \min_{y \in A_2} \max_{x \in A_1} u_1(x, y). \quad (2)$$

## Proof.

For any real-valued function  $f$ , we have

$$\min_z -f(z) = - \max_z f(z). \quad (3)$$

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Proof (ctd.)

Thus, for all  $y \in A_2$ ,

$$\begin{aligned} - \min_{y \in A_2} \max_{x \in A_1} u_1(x, y) &\stackrel{(3)}{=} \max_{y \in A_2} - \max_{x \in A_1} u_1(x, y) \\ &\stackrel{(3)}{=} \max_{y \in A_2} \min_{x \in A_1} -u_1(x, y) \\ &\stackrel{ZS}{=} \max_{y \in A_2} \min_{x \in A_1} u_2(x, y). \end{aligned}$$

□

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



Now, we are ready to prove our  
main theorem about zero-sum games and Nash equilibria.

In zero-sum games:

- 1 Every Nash equilibrium is a pair of maximinimizers.
- 2 All Nash equilibria have the same payoffs.
- 3 If there is at least one Nash equilibrium, then every pair of maximinimizers is a Nash equilibrium.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Theorem (Maximinimizer theorem)

Let  $G = (\{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N})$  be a zero-sum game. Then:

- 1 If  $(x^*, y^*)$  is a Nash equilibrium of  $G$ , then  $x^*$  and  $y^*$  are maximinimizers for player 1 and player 2, respectively.
- 2 If  $(x^*, y^*)$  is a Nash equilibrium of  $G$ , then

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y) = u_1(x^*, y^*).$$

- 3 If  $\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$ , and  $x^*$  and  $y^*$  maximinimizers of player 1 and player 2 respectively, then  $(x^*, y^*)$  is a Nash equilibrium.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Proof.

1 Let  $(x^*, y^*)$  be a Nash equilibrium. Then

$$u_2(x^*, y^*) \geq u_2(x^*, y) \quad \text{for all } y \in A_2.$$

With  $u_1 = -u_2$ , this implies

$$u_1(x^*, y^*) \leq u_1(x^*, y) \quad \text{for all } y \in A_2.$$

Thus

$$u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y) \leq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \quad (4)$$

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Proof (ctd.)

1 (ctd.)

Furthermore, since  $(x^*, y^*)$  is a Nash equilibrium, also

$$u_1(x^*, y^*) \geq u_1(x, y^*) \quad \text{for all } x \in A_1.$$

Hence

$$u_1(x^*, y^*) \geq \max_{x \in A_1} u_1(x, y^*).$$

This implies

$$u_1(x^*, y^*) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \quad (5)$$

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Proof (ctd.)

1 (ctd.)

Inequalities (4) and (5) together imply that

$$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \quad (6)$$

Thus,  $x^*$  is a maximinimizer for player 1.

Similarly, we can show that  $y^*$  is a maximinimizer for player 2:

$$u_2(x^*, y^*) = \max_{y \in A_2} \min_{x \in A_1} u_2(x, y). \quad (7)$$

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Proof (ctd.)

2 We only need to put things together:

$$\begin{aligned} \max_{x \in A_1} \min_{y \in A_2} u_1(x, y) &\stackrel{(6)}{=} u_1(x^*, y^*) \\ &\stackrel{\text{ZS}}{=} -u_2(x^*, y^*) \\ &\stackrel{(7)}{=} -\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) \\ &\stackrel{(2)}{=} \min_{y \in A_2} \max_{x \in A_1} u_1(x, y). \end{aligned}$$

In particular, it follows that all Nash equilibria share the same payoff profile.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

## Proof (ctd.)

- 3 Let  $x^*$  and  $y^*$  be maximinimizers for player 1 and 2, respectively, and assume that

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y) =: v^*. \quad (8)$$

With Equation (2) from the previous lemma, we get

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) = -v^*. \quad (9)$$

With  $x^*$  and  $y^*$  being maximinimizers, (8) and (9) imply

$$u_1(x^*, y) \geq v^* \quad \text{for all } y \in A_2, \text{ and} \quad (10)$$

$$u_2(x, y^*) \geq -v^* \quad \text{for all } x \in A_1. \quad (11)$$

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



## Proof (ctd.)

3 (ctd.)

Special cases of (10) and (11) for  $x = x^*$  and  $y = y^*$ :

$$u_1(x^*, y^*) \geq v^* \quad \text{and} \quad u_2(x^*, y^*) \geq -v^*.$$

With  $u_1 = -u_2$ , the latter is equivalent to  $u_1(x^*, y^*) \leq v^*$ , which gives us

$$u_1(x^*, y^*) = v^*. \quad (12)$$

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



## Proof (ctd.)

3 (ctd.)

Plugging (12) into the right-hand side of (10) gives us

$$u_1(x^*, y) \geq u_1(x^*, y^*) \quad \text{for all } y \in A_2.$$

With  $u_1 = -u_2$ , this is equivalent to

$$u_2(x^*, y) \leq u_2(x^*, y^*) \quad \text{for all } y \in A_2.$$

In other words,  $y^*$  is a best response to  $x^*$ .

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



## Proof (ctd.)

3 (ctd.)

Similarly, we can plug (12) into the right-hand side of (11) and obtain

$$u_2(x, y^*) \geq -u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

Again using  $u_1 = -u_2$ , this is equivalent to

$$u_1(x, y^*) \leq u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

In words,  $x^*$  is also a best response to  $y^*$ .

Hence,  $(x^*, y^*)$  is a Nash equilibrium.



Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



## Corollary

Let  $G = \langle \{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a zero-sum game, and let  $(x_1^*, y_1^*)$  and  $(x_2^*, y_2^*)$  be two Nash equilibria of  $G$ .

Then  $(x_1^*, y_2^*)$  and  $(x_2^*, y_1^*)$  are also Nash equilibria of  $G$ .

**In other words:** Nash equilibria of zero-sum games can be arbitrarily recombined.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary



## Proof.

With part (1) of the maximinimizer theorem, we get that  $x_1^*$  and  $x_2^*$  are maximinimizers for player 1 and that  $y_1^*$  and  $y_2^*$  are maximinimizers for player 2.

With part (2) of the maximinimizer theorem, we get that  $\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$ .

With this equality, with  $x_1^*$ ,  $x_2^*$ ,  $y_1^*$ , and  $y_2^*$  all being maximinimizers, and with part (3) of the maximinimizer theorem, we get that  $(x_1^*, y_2^*)$  and  $(x_2^*, y_1^*)$  are also Nash equilibria of  $G$ . □

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary

# 6 Summary



Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

**Summary**



- **Strategic games** are one-shot games of finitely many players with given action sets and payoff functions. Players have perfect information.
- **Solution concepts:** survival of **iterative elimination of strictly dominated strategies**, **Nash equilibria**.
- **Relation between solution concepts:** Nash equilibria always survive iterative elimination of strictly dominated strategies.
- In **zero-sum games**, one player's gain is the other player's loss. Thus, playing it safe is rational. Relevant concept: **maximinimizers**.
- **Relation to Nash equilibria:** In zero-sum games, Nash equilibria are pairs of maximinimizers, and, if at least one Nash equilibrium exists, pairs of maximinimizers are also Nash equilibria.

Preliminaries  
and  
Examples

Solution  
Concepts  
and Notation

Dominated  
Strategies

Nash  
Equilibria

Zero-Sum  
Games

Summary