

# Game Theory

## 10. Mechanisms Without Money

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June 28th, 2017

# 1 Motivation



Motivation

House  
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# Mechanisms without Money



## Motivation 1:

- According to Gibbard-Satterthwaite:  
In general, **nontrivial social choice functions manipulable**.
- **One way out: Introduction of money**  
(cf. VCG mechanisms)
- **Other way out: Restriction of preferences**  
(cf. single-peaked preferences; this chapter)

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## Motivation 2:

- Introduction of central concept from cooperative game theory: **the core**

## Examples:

- House allocation problem
- Stable matchings

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# 2 House Allocation Problem



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- Top Trading Cycle Algorithm

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- **Players**  $N = \{1, \dots, n\}$ .
- Each player  $i$  owns house  $i$ .
- Each player  $i$  has **strict linear preference** order  $\prec_i$  over the set of houses.  
**Example:**  $j \prec_i k$  means player  $i$  prefers house  $k$  to house  $j$ .
- **Alternatives**  $A$ : allocations of houses to players (permutations  $\pi \in S_n$  of  $N$ ).  
**Example:**  $\pi(i) = j$  means player  $i$  gets house  $j$ .
- **Objective:** **reallocate the houses** among the agents “appropriately”.

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- **Note on preference relations:**
  - Arbitrary (strict linear) preference orders  $\prec_i$  over houses,
  - but **no** arbitrary preference orders  $\preceq_i$  over  $A$ .
- **Rather:** Player  $i$  **indifferent** between different allocations  $\pi_1$  and  $\pi_2$  as long as  $\pi_1(i) = \pi_2(i)$ .  
Indifference denoted as  $\pi_1 \approx_i \pi_2$ .
- If player  $i$  is not indifferent:  $\pi_1 \prec_i \pi_2$  iff  $\pi_1(i) \prec_i \pi_2(i)$ .
- **Notation:**  $\pi_1 \preceq_i \pi_2$  iff  $\pi_1 \prec_i \pi_2$  or  $\pi_1 \approx_i \pi_2$ .
- This makes **Gibbard-Satterthwaite inapplicable**.

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- **Important new aspect** of house allocation problem:  
**players control resources** to be allocated.
- **Allocation can be subverted by subset of agents breaking away** and trading among themselves.
- How to **avoid** such allocations?
- How to make allocation mechanism **non-manipulable**?

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**Notation:** For  $M \subseteq N$ , let

$$A(M) = \{\pi \in A \mid \forall i \in M : \pi(i) \in M\}$$

be the set of allocations that can be achieved by the agents in  $M$  trading among themselves.

**Definition (blocking coalition)**

Let  $\pi \in A$  be an allocation. A set  $M \subseteq N$  is called a **blocking coalition** for  $\pi$  if there exists a  $\pi' \in A(M)$  such that

- $\pi \preceq_i \pi'$  for all  $i \in M$  and
- $\pi \prec_i \pi'$  for at least one  $i \in M$ .

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# House Allocation Problem



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## Intuition:

A blocking coalition can receive houses everyone from the coalition likes at least as much as under allocation  $\pi$ , with at least one player being strictly better off, by trading among themselves.

## Definition (core)

The set of allocations that is not blocked by any subset of agents is called the **core**.

**Question:** Is the core nonempty?

# Top Trading Cycle Algorithm (TTCA)



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- Algorithm to construct allocation
- Let  $G = \langle V, A, c \rangle$  be an arc-colored directed graph where:
  - $V = N$  (i.e., one vertex for each player),
  - $A = V \times V$ , and
  - $c : A \rightarrow N$  such that  $c(i, j) = k$  if house  $j$  is player  $i$ 's  $k$ th ranked choice according to  $\triangleleft_i$ .
- **Note:** Loops  $(i, i)$  are allowed. We treat them as cycles of length 0.

# Top Trading Cycle Algorithm (TTCA)



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## Pseudocode:

```

let  $\pi(i) = i$  for all  $i \in N$ .
while players unaccounted for do
    consider subgraph  $G'$  of  $G$  where each vertex has
        only one outgoing arc: the least-colored one from  $G$ .
    identify cycles in  $G'$ .
    add corresponding cyclic permutations to  $\pi$ .
    delete players accounted for and incident edges from  $G$ .
end while
output  $\pi$ .
    
```

## Notation:

Let  $N_i$  be the set of vertices on cycles identified in iteration  $i$ .

# Top Trading Cycle Algorithm (TTCA)

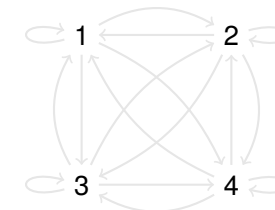


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## Example:

- **Player 1:**  $3 \triangleleft_1 1 \triangleleft_1 4 \triangleleft_1 2$
- **Player 2:**  $4 \triangleleft_2 2 \triangleleft_2 3 \triangleleft_2 1$
- **Player 3:**  $3 \triangleleft_3 4 \triangleleft_3 2 \triangleleft_3 1$
- **Player 4:**  $1 \triangleleft_4 4 \triangleleft_4 2 \triangleleft_4 3$

## Corresponding graph:



- **Iteration 1:**  $\pi(1) = 2, \pi(2) = 1$ .
- **Iteration 2:**  $\pi(3) = 4, \pi(4) = 3$ .
- **Done:**  $\pi(1) = 2, \pi(2) = 1, \pi(3) = 4, \pi(4) = 3$ .

# Top Trading Cycle Algorithm (TTCA)



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## Theorem

The core of the house allocation problem consists of exactly one matching.

## Proof sketch

**At most one matching:** Show that if a matching is in the core, it must be the one returned by the TTCA.

In TTCA, each player in  $N_1$  receives his favorite house.

Therefore,  $N_1$  would form a blocking coalition to any allocation that does not assign to all of those players the houses they would receive in TTCA.

...

# Top Trading Cycle Algorithm (TTCA)



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## Proof sketch (ctd.)

That is, any core allocation must assign  $N_1$  to houses as TTCA assigns them.

Argument can be extended inductively to  $N_k$ ,  $2 \leq k \leq n$ .

**At least one matching:** Show that TTCA allocation is in the core, i.e., that there is no other blocking coalition  $M \subseteq N$ .  
Homework. □

# Top Trading Cycle Mechanism (TTCM)



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**Question:** What about manipulability?

## Definition (top trading cycle mechanism)

The **top trading cycle mechanism (TTCM)** is the function that, for each profile of preferences, returns the allocation computed by the TTCA.

## Theorem

The TTCM cannot be manipulated.

## Proof

Homework. □

# 3 Stable Matchings



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- Definitions
- Deferred Acceptance Algorithm
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## Problem statement:

- Given disjoint finite sets  $M$  of men and  $W$  of women.
- Assume WLOG that  $|M| = |W|$  (introduce dummy-men/dummy-women).
- Each  $m \in M$  has strict preference ordering  $\prec_m$  over  $W$ .
- Each  $w \in W$  has strict preference ordering  $\prec_w$  over  $M$ .
- **Matching**: “appropriate” assignment of men to women such that each man is assigned to at most one woman and vice versa.

Note: A group of players can **subvert a matching** by opting out.

## Definition (stability, blocking pair)

A matching is called **unstable** if there are two men  $m, m'$  and two women  $w, w'$  such that

- $m$  is matched to  $w$ ,
- $m'$  is matched to  $w'$ , and
- $w \prec_m w'$  and  $m' \prec_{w'} m$ .

The pair  $\langle m, w' \rangle$  is called a **blocking pair**.

A matching that has no blocking pairs is called **stable**.

## Definition (core)

The **core** of the matching game is the set of all stable matchings.

## Example:

- Man 1:  $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2:  $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3:  $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1:  $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2:  $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3:  $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

## Two matchings:

- Matching  $\{\langle m_1, w_1 \rangle, \langle m_2, w_2 \rangle, \langle m_3, w_3 \rangle\}$ 
  - unstable ( $\langle m_1, w_2 \rangle$  is a blocking pair)
- Matching  $\{\langle m_1, w_1 \rangle, \langle m_3, w_2 \rangle, \langle m_2, w_3 \rangle\}$ 
  - stable

Question: Is there always a stable matching?

Answer: Yes! And it can even be efficiently constructed.

How? Deferred acceptance algorithm!

## Definition (deferred acceptance algorithm, male proposals)

- 1 Each man proposes to his top-ranked choice.
- 2 Each woman who has received at least one proposal (including tentatively kept one from earlier rounds) tentatively keeps top-ranked proposal and rejects rest.
- 3 If no man is left rejected, stop.
- 4 Otherwise, each man who has been rejected proposes to his top-ranked choice among the women who have not rejected him. Then, goto 2.

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## Note:

- Algorithm has polynomial runtime.
- No man is assigned to more than one woman.
- No woman is assigned to more than one man.
- $\rightsquigarrow$  matching

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## Example:

- Man 1:  $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2:  $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3:  $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1:  $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2:  $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3:  $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

## Deferred acceptance algorithm:

- 1  $m_1$  proposes to  $w_2$ ,  $m_2$  to  $w_1$ , and  $m_3$  to  $w_1$ .
- 2  $w_1$  keeps  $m_3$  and rejects  $m_2$ ,  $w_2$  keeps  $m_1$ .
- 3  $m_2$  now proposes to  $w_3$ .
- 4  $w_3$  keeps  $m_2$ .

Resulting matching:  $\{\langle m_1, w_2 \rangle, \langle m_2, w_3 \rangle, \langle m_3, w_1 \rangle\}$ .

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## Theorem

The deferred acceptance algorithm with male proposals terminates in a stable matching.

## Proof

Suppose not.

Then there exists a blocking pair  $\langle m_1, w_1 \rangle$  with  $m_1$  matched to some  $w_2$  and  $w_1$  matched to some  $m_2$ .

Since  $\langle m_1, w_1 \rangle$  is blocking and  $w_2 \prec_{m_1} w_1$ , in the proposal algorithm,  $m_1$  would have proposed to  $w_1$  before  $w_2$ .

Since  $m_1$  was not matched with  $w_1$  by the algorithm, it must be because  $w_1$  received a proposal from a man she ranked higher than  $m_1$ . ...

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## Proof (ctd.)

Since the algorithm matches her to  $m_2$  it follows that  $m_1 \prec_{w_1} m_2$ .

This contradicts the fact that  $\langle m_1, w_1 \rangle$  is a blocking pair.  $\square$

Analogous version where the women propose: outcome would also be a stable matching.

Denote a matching by  $\mu$ . The woman assigned to man  $m$  in  $\mu$  is  $\mu(m)$ , and the man assigned to woman  $w$  is  $\mu(w)$ .

## Definition (optimality)

A matching  $\mu$  is **male-optimal** if there is no stable matching  $\nu$  such that  $\mu(m) \prec_m \nu(m)$  or  $\mu(m) = \nu(m)$  for all  $m \in M$  and  $\mu(m) \prec_m \nu(m)$  for at least one  $m \in M$ . **Female-optimal**: similar.

## Theorem

- The stable matching produced by the (fe)male-proposal deferred acceptance algorithm is (fe)male-optimal.
- In general, there is no stable matching that is male-optimal and female-optimal.  $\square$

## Theorem

The mechanism associated with the (fe)male-proposal algorithm cannot be manipulated by the (fe)males.  $\square$

- **Avoid Gibbard-Satterthwaite** by restricting domain of preferences.
- **House allocation** problem:
  - Solved using **top trading cycle** algorithm.
  - Algorithm finds **unique solution in the core**, where no **blocking coalition** of players has an incentive to break away.
  - The top trading cycle mechanism **cannot be manipulated**.
- **Stable matchings**:
  - Solved using **deferred acceptance** algorithm.
  - Algorithm finds **a stable matching in the core**, where no **blocking pair** of players has an incentive to break away.
  - The mechanism associated with the (fe)male-proposal algorithm cannot be manipulated by the (fe)males.

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