Mixed Strategies
Mixed Strategies

**Observation:** Not every strategic game has a pure-strategy Nash equilibrium (e.g. matching pennies).

**Question:**
- Can we do anything about that?
- Which strategy to play then?

**Idea:** Consider randomized strategies.
Mixed Strategies

Notation

Let $X$ be a set.

Then $\Delta(X)$ denotes the set of probability distributions over $X$.

That is, each $p \in \Delta(X)$ is a mapping $p : X \rightarrow [0, 1]$ with

$$\sum_{x \in X} p(x) = 1.$$
A mixed strategy is a strategy where a player is allowed to randomize his action (throw a dice mentally and then act according to what he has decided to do for each outcome).

**Definition (Mixed strategy)**

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A **mixed strategy** of player $i$ in $G$ is a probability distribution $\alpha_i \in \Delta(A_i)$ over player $i$’s actions.

For $a_i \in A_i$, $\alpha_i(a_i)$ is the probability for playing $a_i$.

**Terminology:** When we talk about strategies in $A_i$ specifically, to distinguish them from mixed strategies, we sometimes also call them **pure strategies**.
Mixed Strategies

Definition (Mixed strategy profile)

A profile \( \alpha = (\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i) \) of mixed strategies induces a probability distribution \( p_\alpha \) over \( A = \prod_{i \in N} A_i \) as follows:

\[
p_\alpha(a) = \prod_{i \in N} \alpha_i(a_i).
\]

For \( A' \subseteq A \), we define

\[
p_\alpha(A') = \sum_{a \in A'} p_\alpha(a) = \sum_{a \in A'} \prod_{i \in N} \alpha_i(a_i).
\]
Mixed Strategies

Notation

Since each pure strategy \( a_i \in A_i \) is equivalent to its induced mixed strategy \( \hat{a}_i \)

\[
\hat{a}_i(a'_i) = \begin{cases} 
1 & \text{if } a'_i = a_i \\
0 & \text{otherwise}, 
\end{cases}
\]

we sometimes abuse notation and write \( a_i \) instead of \( \hat{a}_i \).
Example (Mixed strategies for matching pennies)

\[
\begin{array}{c|cc}
 & H & T \\
\hline
H & 1, -1 & -1, 1 \\
T & -1, 1 & 1, -1 \\
\end{array}
\]

\[\alpha = (\alpha_1, \alpha_2), \quad \alpha_1(H) = \frac{2}{3}, \quad \alpha_1(T) = \frac{1}{3}, \quad \alpha_2(H) = \frac{1}{3}, \quad \alpha_2(T) = \frac{2}{3}.\]

This induces a probability distribution over \(\{H, T\} \times \{H, T\}\):

\[
\begin{align*}
p_\alpha(H, H) &= \alpha_1(H) \cdot \alpha_2(H) = \frac{2}{9}, & u_1(H, H) &= +1, \\
p_\alpha(H, T) &= \alpha_1(H) \cdot \alpha_2(T) = \frac{4}{9}, & u_1(H, T) &= -1, \\
p_\alpha(T, H) &= \alpha_1(T) \cdot \alpha_2(H) = \frac{1}{9}, & u_1(T, H) &= -1, \\
p_\alpha(T, T) &= \alpha_1(T) \cdot \alpha_2(T) = \frac{2}{9}, & u_1(T, T) &= +1.
\end{align*}
\]
Expected Utility

**Definition (Expected utility)**

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile.

The **expected utility** of $\alpha$ for player $i$ is

$$U_i(\alpha) = U_i\left((\alpha_j)_{j \in N}\right) := \sum_{a \in A} p_\alpha(a) u_i(a) = \sum_{a \in A} \left( \prod_{j \in N} \alpha_j(a_j) \right) u_i(a).$$

**Example (Mixed strategies for matching pennies (ctd.))**

The expected utilities for player 1 and player 2 are

$$U_1(\alpha_1, \alpha_2) = -\frac{1}{9} \quad \text{and} \quad U_2(\alpha_1, \alpha_2) = +\frac{1}{9}.$$
Remark: The expected utility functions $U_i$ are linear in all mixed strategies.

**Proposition**

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile, $\beta_i, \gamma_i \in \Delta(A_i)$ mixed strategies, and $\lambda \in [0, 1]$. Then

$$U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i) = \lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i).$$

Moreover,

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) \cdot U_i(\alpha_{-i}, a_i)$$

**Proof.**

Homework.
Definition (Mixed extension)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. The mixed extension of $G$ is the game $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$, where

- $\Delta(A_i)$ is the set of probability distributions over $A_i$ and
- $U_i : \prod_{j \in N} \Delta(A_j) \rightarrow \mathbb{R}$ assigns to each mixed strategy profile $\alpha$ the expected utility for player $i$ according to the induced probability distribution $p_{\alpha}$. 
Definition (Nash equilibrium in mixed strategies)

Let $G$ be a strategic game.

A *Nash equilibrium in mixed strategies* (or *mixed-strategy Nash equilibrium*) of $G$ is a Nash equilibrium in the mixed extension of $G$. 
Support

Intuition:
- It does not make sense to assign positive probability to a pure strategy that is not a best response to what the other players do.
- **Claim:** A profile of mixed strategies $\alpha$ is a Nash equilibrium if and only if everyone only plays best pure responses to what the others play.

**Definition (Support)**

Let $\alpha_i$ be a mixed strategy.

The **support** of $\alpha_i$ is the set

$$supp(\alpha_i) = \{ a_i \in A_i \mid \alpha_i(a_i) > 0 \}$$

of actions played with nonzero probability.
Support Lemma

Lemma (Support lemma)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game.

Then $\alpha^* \in \prod_{i \in N} \Delta(A_i)$ is a mixed-strategy Nash equilibrium in $G$ if and only if for every player $i \in N$, every pure strategy in the support of $\alpha^*_i$ is a best response to $\alpha^*_{-i}$.

For a single player—given all other players stick to their mixed strategies—it does not make a difference whether he plays the mixed strategy or whether he plays any single pure strategy from the support of the mixed strategy.
Example (Support lemma)

Matching pennies, strategy profile $\alpha = (\alpha_1, \alpha_2)$ with

$$\alpha_1(H) = \frac{2}{3}, \quad \alpha_1(T) = \frac{1}{3}, \quad \alpha_2(H) = \frac{1}{3}, \text{ and } \alpha_2(T) = \frac{2}{3}.$$ 

For $\alpha$ to be a Nash equilibrium, both actions in $\text{supp}(\alpha_2) = \{H, T\}$ have to be best responses to $\alpha_1$. Are they?

$$U_2(\alpha_1, H) = \alpha_1(H) \cdot u_2(H, H) + \alpha_1(T) \cdot u_2(T, H)$$
$$= \frac{2}{3} \cdot (-1) + \frac{1}{3} \cdot (+1) = -\frac{1}{3},$$

$$U_2(\alpha_1, T) = \alpha_1(H) \cdot u_2(H, T) + \alpha_1(T) \cdot u_2(T, T)$$
$$= \frac{2}{3} \cdot (+1) + \frac{1}{3} \cdot (-1) = \frac{1}{3}.$$ 

\[ \Rightarrow \]

Support lemma

$H \in \text{supp}(\alpha_2)$, but $H \notin B_2(\alpha_1)$.

$\alpha$ can not be a Nash equilibrium.
**Support Lemma**

**Proof.**

"⇒": Let \( \alpha^* \) be a Nash equilibrium with \( a_i \in \text{supp}(\alpha_i^*) \).

Assume that \( a_i \) is not a best response to \( \alpha_{-i}^* \). Because \( U_i \) is linear, player \( i \) can improve his utility by shifting probability in \( \alpha_i^* \) from \( a_i \) to a better response.

This makes the modified \( \alpha_i^* \) a better response than the original \( \alpha_i^* \), i.e., the original \( \alpha_i^* \) was not a best response, which contradicts the assumption that \( \alpha^* \) is a Nash equilibrium.
Proof.

“⇒”: Let $\alpha^*$ be a Nash equilibrium with $a_i \in \text{supp}(\alpha_i^*)$. Assume that $a_i$ is not a best response to $\alpha_{-i}^*$. Because $U_i$ is linear, player $i$ can improve his utility by shifting probability in $\alpha_i^*$ from $a_i$ to a better response.

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Support Lemma

Proof (ctd.)

“⇐”: Assume that $\alpha^*$ is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy $\alpha'_i$ such that $U_i(\alpha^*_{-i}, \alpha'_i) > U_i(\alpha^*_{-i}, \alpha^*_i)$.

Because $U_i$ is linear, there must be a pure strategy $a'_i \in \text{supp}(\alpha'_i)$ that has higher utility than some pure strategy $a''_i \in \text{supp}(\alpha^*_i)$.

Therefore, $\text{supp}(\alpha^*_i)$ does not only contain best responses to $\alpha^*_{-i}$.
Support Lemma

Proof (ctd.)

“⇐”: Assume that $\alpha^*$ is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy $\alpha'_i$ such that $U_i(\alpha^*_{-i}, \alpha'_i) > U_i(\alpha^*_{-i}, \alpha^*_i)$.

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Proof (ctd.)

“⇐”: Assume that $\alpha^*$ is not a Nash equilibrium.

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“⇐”: Assume that $\alpha^*$ is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy $\alpha'_i$ such that $U_i(\alpha^*_{-i}, \alpha'_i) > U_i(\alpha^*_{-i}, \alpha^*_i)$.

Because $U_i$ is linear, there must be a pure strategy $a'_i \in supp(\alpha'_i)$ that has higher utility than some pure strategy $a''_i \in supp(\alpha^*_i)$.

Therefore, $supp(\alpha^*_i)$ does not only contain best responses to $\alpha^*_i$. 

□
We already know: $(B, B)$ and $(S, S)$ are pure Nash equilibria. Possible supports (excluding “pure-vs-pure” strategies) are:

- $\{B\}$ vs. $\{B, S\}$
- $\{S\}$ vs. $\{B, S\}$
- $\{B, S\}$ vs. $\{B\}$
- $\{B, S\}$ vs. $\{S\}$
- $\{B, S\}$ vs. $\{B, S\}$

Observation: In Bach or Stravinsky, pure strategies have unique best responses. Therefore, there can be no Nash equilibria of “pure-vs-strictly-mixed” type.
Consequence: Only need to search for additional Nash equilibria with support sets \( \{B, S\} \) vs. \( \{B, S\} \).

Assume that \( (\alpha_1^*, \alpha_2^*) \) is a Nash equilibrium with \( 0 < \alpha_1^*(B) < 1 \) and \( 0 < \alpha_2^*(B) < 1 \). Then

\[
U_1(B, \alpha_2^*) = U_1(S, \alpha_2^*)
\]

\[
\Rightarrow 2 \cdot \alpha_2^*(B) + 0 \cdot \alpha_2^*(S) = 0 \cdot \alpha_2^*(B) + 1 \cdot \alpha_2^*(S)
\]

\[
\Rightarrow 2 \cdot \alpha_2^*(B) = 1 - \alpha_2^*(B)
\]

\[
\Rightarrow 3 \cdot \alpha_2^*(B) = 1
\]

\[
\Rightarrow \alpha_2^*(B) = \frac{1}{3} \quad \text{(and} \quad \alpha_2^*(S) = \frac{2}{3})
\]

Similarly, we get \( \alpha_1^*(B) = \frac{2}{3} \) and \( \alpha_1^*(S) = \frac{1}{3} \).

The payoff profile of this equilibrium is \( (\frac{2}{3}, \frac{2}{3}) \).
Let $G = \langle \{1, 2\}, (A_i), (u_i) \rangle$ with $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$ be a two-player game with two actions each, and $(T, \alpha^*_2)$ with $0 < \alpha^*_2(L) < 1$ be a Nash equilibrium of $G$.

Then at least one of the profiles $(T, L)$ and $(T, R)$ is also a Nash equilibrium of $G$.

**Reason:** Both $L$ and $R$ are best responses to $T$. Assume that $T$ was neither a best response to $L$ nor to $R$. Then $B$ would be a better response than $T$ both to $L$ and to $R$.

With the linearity of $U_1$, $B$ would also be a better response to $\alpha^*_2$ than $T$ is. Contradiction.
Let $G = \langle \{1, 2\}, (A_i), (u_i) \rangle$ with $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$ be a two-player game with two actions each, and $(T, \alpha_2^*)$ with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of $G$.

Then at least one of the profiles $(T, L)$ and $(T, R)$ is also a Nash equilibrium of $G$.

**Reason:** Both $L$ and $R$ are best responses to $T$. Assume that $T$ was neither a best response to $L$ nor to $R$. Then $B$ would be a better response than $T$ both to $L$ and to $R$.

With the linearity of $U_1$, $B$ would also be a better response to $\alpha_2^*$ than $T$ is. Contradiction.
Support Lemma

Remark

Let $G = \langle \{1, 2\}, (A_i), (u_i) \rangle$ with $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$ be a two-player game with two actions each, and $(T, \alpha_2^*)$ with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of $G$.

Then at least one of the profiles $(T, L)$ and $(T, R)$ is also a Nash equilibrium of $G$.

**Reason:** Both $L$ and $R$ are best responses to $T$. Assume that $T$ was neither a best response to $L$ nor to $R$. Then $B$ would be a better response than $T$ both to $L$ and to $R$.

With the linearity of $U_1$, $B$ would also be a better response to $\alpha_2^*$ than $T$ is. Contradiction.
Consider the Nash equilibrium $\alpha^* = (\alpha_1^*, \alpha_2^*)$ with

$$\begin{align*}
\alpha_1^*(T) &= 1, & \alpha_1^*(B) &= 0, & \alpha_2^*(L) &= \frac{1}{10}, & \alpha_2^*(R) &= \frac{9}{10}
\end{align*}$$

in the following game:

$$\begin{array}{c|cc}
 & L & R \\
\hline
T & 1, 1 & 1, 1 \\
B & 2, 2 & -5, -5 \\
\end{array}$$

Here, $(T, R)$ is also a Nash equilibrium.
Nash’s Theorem
Motivation: When does a strategic game have a mixed-strategy Nash equilibrium?

In the previous chapter, we discussed necessary and sufficient conditions for the existence of Nash equilibria for the special case of zero-sum games. Can we make other claims?
Nash’s Theorem

Theorem (Nash’s theorem)

Every finite strategic game has a mixed-strategy Nash equilibrium.

Proof sketch.

Consider the set-valued function of best responses

\[ B : \mathbb{R}^{\sum_i |A_i|} \rightarrow 2^{\mathbb{R}^{\sum_i |A_i|}} \]

with

\[ B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i}). \]

A mixed strategy profile \( \alpha \) is a fixed point of \( B \) if and only if \( \alpha \in B(\alpha) \) if and only if \( \alpha \) is a mixed-strategy Nash equilibrium. The graph of \( B \) has to be connected. Then there is at least one point on the fixpoint diagonal.
Nash’s Theorem

Outline for the formal proof:

1. Review of necessary mathematical definitions
   ⇝ Subsection “Definitions”

2. Statement of a fixpoint theorem used to prove Nash’s theorem (without proof)
   ⇝ Subsection “Kakutani’s Fixpoint Theorem”

3. Proof of Nash’s theorem using fixpoint theorem
   ⇝ Subsection “Proof of Nash’s Theorem”
A set $X \subseteq \mathbb{R}^n$ is **closed** if $X$ contains all its limit points, i.e., if $(x_k)_{k \in \mathbb{N}}$ is a sequence of elements in $X$ and $\lim_{k \to \infty} x_k = x$, then also $x \in X$. 

**Example**

- **Closed:**
  - $x \in X$

- **Not closed:**
  - $x \notin X$
Definition

A set $X \subseteq \mathbb{R}^n$ is **bounded** if for each $i = 1, \ldots, n$ there are lower and upper bounds $a_i, b_i \in \mathbb{R}$ such that

$$X \subseteq \prod_{i=1}^{n} [a_i, b_i].$$

Example

**Bounded:**

**Not bounded:**
A set $X \subseteq \mathbb{R}^n$ is convex if for all $x, y \in X$ and all $\lambda \in [0, 1],$

$$\lambda x + (1 - \lambda)y \in X.$$
Definition

For a function $f : X \to 2^X$, the graph of $f$ is the set

$$\text{Graph}(f) = \{(x,y) \mid x \in X, \ y \in f(x)\}.$$
Theorem (Kakutani’s fixpoint theorem)

Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed, bounded and convex set and let $f : X \rightarrow 2^X$ be a function such that

- for all $x \in X$, the set $f(x) \subseteq X$ is nonempty and convex, and
- $\text{Graph}(f)$ is closed.

Then there is an $x \in X$ with $x \in f(x)$, i.e., $f$ has a fixpoint.

Proof.

See Shizuo Kakutani, A generalization of Brouwer’s fixed point theorem, 1941, or your favorite advanced calculus textbook, or the Internet.

For German speakers: Harro Heuser, Lehrbuch der Analysis, Teil 2, also has a proof (Abschnitt 232).
Nash’s Theorem
Kakutani’s Fixpoint Theorem

Example

Let $X = [0, 1]$.

Kakutani’s theorem applicable:

Kakutani’s theorem not applicable:

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Proof.

Apply Kakutani’s fixpoint theorem using $X = \mathcal{A} = \prod_{i \in N} \Delta(A_i)$ and $f = B$, where $B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i})$.

We have to show:

1. $\mathcal{A}$ is nonempty,
2. $\mathcal{A}$ is closed,
3. $\mathcal{A}$ is bounded,
4. $\mathcal{A}$ is convex,
5. $B(\alpha)$ is nonempty for all $\alpha \in \mathcal{A}$,
6. $B(\alpha)$ is convex for all $\alpha \in \mathcal{A}$, and
7. $\text{Graph}(B)$ is closed.
Proof (ctd.)

Some notation:

- Assume without loss of generality that $N = \{1, \ldots, n\}$.
- A profile of mixed strategies can be written as a vector of $M = \sum_{i \in N} |A_i|$ real numbers in the interval $[0, 1]$ such that numbers for the same player add up to 1.

For example, $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1(T) = 0.7$, $\alpha_1(M) = 0.0$, $\alpha_1(B) = 0.3$, $\alpha_2(L) = 0.4$, $\alpha_2(R) = 0.6$ can be seen as the vector

\[(0.7, 0.0, 0.3, 0.4, 0.6)\]

- This allows us to interpret the set $A$ of mixed strategy profiles as a subset of $\mathbb{R}^M$. 

Nash’s Theorem

Proof (ctd.)

1. **Nonempty**: Trivial. \( \mathcal{A} \) contains the tuple

\[
(1, 0, \ldots, 0, \ldots, 1, 0, \ldots, 0).
\]

- \( |A_1| \) times
- \( |A_n| \) times

2. **Closed**: Let \( \alpha_1, \alpha_2, \ldots \) be a sequence in \( \mathcal{A} \) that converges to \( \lim_{k \to \infty} \alpha_k = \alpha \). Suppose \( \alpha \notin \mathcal{A} \). Then either there is some component of \( \alpha \) that is less than zero or greater than one, or the components for some player \( i \) add up to a value other than one.

Since \( \alpha \) is a limit point, the same must hold for some \( \alpha_k \) in the sequence. But then, \( \alpha_k \notin \mathcal{A} \), a contradiction. Hence \( \mathcal{A} \) is closed.
Proof (ctd.)

1. **\( \mathcal{A} \) nonempty:** Trivial. \( \mathcal{A} \) contains the tuple

\[
(1, 0, \ldots, 0, \ldots, 1, 0, \ldots, 0).
\]

\(|A_1|\) −1 times \hspace{1cm} |A_n|\) −1 times

2. **\( \mathcal{A} \) closed:** Let \( \alpha_1, \alpha_2, \ldots \) be a sequence in \( \mathcal{A} \) that converges to \( \lim_{k \to \infty} \alpha_k = \alpha \). Suppose \( \alpha \notin \mathcal{A} \). Then either there is some component of \( \alpha \) that is less than zero or greater than one, or the components for some player \( i \) add up to a value other than one.

Since \( \alpha \) is a limit point, the same must hold for some \( \alpha_k \) in the sequence. But then, \( \alpha_k \notin \mathcal{A} \), a contradiction. Hence \( \mathcal{A} \) is closed.
Proof (ctd.)

1. \( \mathcal{A} \) nonempty: Trivial. \( \mathcal{A} \) contains the tuple

\[
(1, 0, \ldots, 0, \ldots, 1, 0, \ldots, 0).
\]

\(|A_1| - 1 \) times \hspace{1cm} \(|A_n| - 1 \) times

2. \( \mathcal{A} \) closed: Let \( \alpha_1, \alpha_2, \ldots \) be a sequence in \( \mathcal{A} \) that converges to \( \lim_{k \to \infty} \alpha_k = \alpha \). Suppose \( \alpha \notin \mathcal{A} \). Then either there is some component of \( \alpha \) that is less than zero or greater than one, or the components for some player \( i \) add up to a value other than one.

Since \( \alpha \) is a limit point, the same must hold for some \( \alpha_k \) in the sequence. But then, \( \alpha_k \notin \mathcal{A} \), a contradiction. Hence \( \mathcal{A} \) is closed.
Proof (ctd.)

3 \( \mathcal{A} \) bounded: Trivial. All entries are between 0 and 1, i.e., \( \mathcal{A} \) is bounded by \([0, 1]^M\).

4 \( \mathcal{A} \) convex: Let \( \alpha, \beta \in \mathcal{A} \) and \( \lambda \in [0, 1] \), and consider
\[
\gamma = \lambda \alpha + (1 - \lambda) \beta.
\]
Then
\[
\min(\gamma) = \min(\lambda \alpha + (1 - \lambda) \beta) \\
\geq \lambda \cdot \min(\alpha) + (1 - \lambda) \cdot \min(\beta) \\
\geq \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0,
\]
and similarly, \( \max(\gamma) \leq 1 \).

Hence, all entries in \( \gamma \) are still in \([0, 1]\).
Nash’s Theorem

Proof (ctd.)

3. **Convexity**

   Let \( \mathcal{A} \) be convex. Let \( \alpha, \beta \in \mathcal{A} \) and \( \lambda \in [0, 1] \), and consider

   \[
   \gamma = \lambda \alpha + (1 - \lambda) \beta.
   \]

   Then

   \[
   \min(\gamma) = \min(\lambda \alpha + (1 - \lambda) \beta) 
   \geq \lambda \cdot \min(\alpha) + (1 - \lambda) \cdot \min(\beta) 
   \geq \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0,
   \]

   and similarly,

   \[
   \max(\gamma) \leq 1.
   \]

   Hence, all entries in \( \gamma \) are still in \([0, 1]\).
Proof (ctd.)

3. **Bounded**: Trivial. All entries are between 0 and 1, i.e., \( A \) is bounded by \([0, 1]^M\).

4. **Convex**: Let \( \alpha, \beta \in A \) and \( \lambda \in [0, 1] \), and consider 
\[
\gamma = \lambda \alpha + (1 - \lambda) \beta.
\]
Then

\[
\min(\gamma) = \min(\lambda \alpha + (1 - \lambda) \beta) \\
\geq \lambda \cdot \min(\alpha) + (1 - \lambda) \cdot \min(\beta) \\
\geq \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0,
\]

and similarly, \( \max(\gamma) \leq 1 \).

Hence, all entries in \( \gamma \) are still in \([0, 1]\).
Nash’s Theorem

Proof

Proof (ctd.)

3 \mathcal{A} \text{ bounded:} Trivial. All entries are between 0 and 1, i.e., \mathcal{A} is bounded by [0, 1]^{M}.

4 \mathcal{A} \text{ convex:} Let \alpha, \beta \in \mathcal{A} and \lambda \in [0, 1], and consider \gamma = \lambda \alpha + (1 - \lambda) \beta. Then

\[
\min(\gamma) = \min(\lambda \alpha + (1 - \lambda) \beta) \\
\geq \lambda \cdot \min(\alpha) + (1 - \lambda) \cdot \min(\beta) \\
\geq \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0,
\]

and similarly, \max(\gamma) \leq 1.

Hence, all entries in \gamma are still in [0, 1].
Proof (ctd.)

4) convex (ctd.): Let \( \tilde{\alpha}, \tilde{\beta} \) and \( \tilde{\gamma} \) be the sections of \( \alpha, \beta \) and \( \gamma \), respectively, that determine the probability distribution for player \( i \). Then

\[
\sum \tilde{\gamma} = \sum (\lambda \tilde{\alpha} + (1 - \lambda) \tilde{\beta}) \\
= \lambda \cdot \sum \tilde{\alpha} + (1 - \lambda) \cdot \sum \tilde{\beta} \\
= \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1.
\]

Hence, all probabilities for player \( i \) in \( \gamma \) still sum up to 1. Altogether, \( \gamma \in A \), and therefore, \( A \) is convex.
Proof (ctd.)

\( \mathcal{A} \) convex (ctd.): Let \( \tilde{\alpha}, \tilde{\beta} \) and \( \tilde{\gamma} \) be the sections of \( \alpha, \beta \) and \( \gamma \), respectively, that determine the probability distribution for player \( i \). Then

\[
\sum \tilde{\gamma} = \sum (\lambda \tilde{\alpha} + (1 - \lambda) \tilde{\beta}) \\
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= \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1.
\]

Hence, all probabilities for player \( i \) in \( \gamma \) still sum up to 1. Altogether, \( \gamma \in \mathcal{A} \), and therefore, \( \mathcal{A} \) is convex.
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\]

\[
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\]

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\[
U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i) = \lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i)
\]

for all \( \lambda \in [0, 1] \).

Hence, \( U_i \) is continuous on \( \Delta(A_i) \).

Continuous functions on closed and bounded sets take their maximum in that set.

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To see this, let \( \alpha_i', \alpha_i'' \in B_i(\alpha_{-i}) \).

Then \( U_i(\alpha_{-i}, \alpha_i') = U_i(\alpha_{-i}, \alpha_i'') \).

With Equation (1), this implies

\[
\lambda \alpha_i' + (1 - \lambda) \alpha_i'' \in B_i(\alpha_{-i}).
\]

Hence, \( B_i(\alpha_{-i}) \) is convex.

7 Graph(\( B \)) closed: Let \( (\alpha^k, \beta^k) \) be a convergent sequence in \( Graph(B) \) with \( \lim_{k \to \infty} (\alpha^k, \beta^k) = (\alpha, \beta) \).

So, \( \alpha^k, \beta^k, \alpha, \beta \in \prod_{i \in N} \Delta(A_i) \) and \( \beta^k \in B(\alpha^k) \).

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Nash’s Theorem
Proof

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We need to show that \((\alpha, \beta) \in \text{Graph}(B)\), i.e., that \(\beta \in B(\alpha)\).
Proof (ctd.)

Graph(B) closed (ctd.): It holds for all \( i \in N \):

\[
U_i(\alpha_{-i}, \beta_i) \overset{(D)}{=} U_i\left( \lim_{k \to \infty} (\alpha_{-i}^k, \beta_i^k) \right)
\]

\[
\overset{(C)}{=} \lim_{k \to \infty} U_i(\alpha_{-i}^k, \beta_i^k)
\]

\[
\overset{(B)}{\geq} \lim_{k \to \infty} U_i(\alpha_{-i}^k, \beta_i') \quad \text{for all } \beta_i' \in \Delta(A_i)
\]

\[
\overset{(C)}{=} U_i\left( \lim_{k \to \infty} \alpha_{-i}^k, \beta_i' \right) \quad \text{for all } \beta_i' \in \Delta(A_i)
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\overset{(D)}{=} U_i(\alpha_{-i}, \beta_i') \quad \text{for all } \beta_i' \in \Delta(A_i).
\]

(D): def. \( \alpha_i, \beta_i \); (C) continuity; (B) \( \beta_i^k \) best response to \( \alpha_{-i}^k \).
Proof (ctd.)

7  \textit{Graph}(B) closed (ctd.):} It follows that $\beta_i$ is a best response to $\alpha_{-i}$ for all $i \in N$.

Thus, $\beta \in B(\alpha)$ and finally $(\alpha, \beta) \in \text{Graph}(B)$.

Therefore, all requirements of Kakutani’s fixpoint theorem are satisfied.

Applying Kakutani’s theorem establishes the existence of a fixpoint of $B$, which is, by definition/construction, the same as a mixed-strategy Nash equilibrium.
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Correlated Equilibria
Correlated Equilibria

Recall: There are three Nash equilibria in Bach or Stravinsky

- \((B, B)\) with payoff profile \((2, 1)\)
- \((S, S)\) with payoff profile \((1, 2)\)
- \((\alpha_1^*, \alpha_2^*)\) with payoff profile \((2/3, 2/3)\) where
  - \(\alpha_1^*(B) = 2/3, \alpha_1^*(S) = 1/3,\)
  - \(\alpha_2^*(B) = 1/3, \alpha_2^*(S) = 2/3.\)

Idea: Use a publicly visible coin toss to decide which action from a mixed strategy is played. This can lead to higher payoffs.
Example (Correlated equilibrium in BoS)

With a fair coin that both players can observe, the players can agree to play as follows:

- If the coin shows heads, both play \( B \).
- If the coin shows tails, both play \( S \).

This is stable in the sense that no player has an incentive to deviate from this agreed-upon rule, as long as the other player keeps playing his/her strategy (cf. definition of Nash equilibria).

Expected payoffs: \( (\frac{3}{2}, \frac{3}{2}) \) instead of \( (\frac{2}{3}, \frac{2}{3}) \).
Observations and Information Partitions

We assume that observations are made based on a finite probability space \((\Omega, \pi)\), where \(\Omega\) is a set of states and \(\pi\) is a probability measure on \(\Omega\).

Agents might not be able to distinguish all states from each other. In order to model this, we assume for each player \(i\) an information partition \(\mathcal{P}_i = \{P_{i1}, P_{i2}, \ldots, P_{ik}\}\). This means that \(\bigcup_{j=1}^{ik} P_j = \Omega\) and for all \(P_i, P_k \in \mathcal{P}_i\) with \(P_j \neq P_k\), we have \(P_j \cap P_k = \emptyset\).

Example: \(\Omega = \{x, y, z\}\), \(\mathcal{P}_1 = \{\{x\}, \{y, z\}\}\), \(\mathcal{P}_2 = \{\{x, y\}, \{z\}\}\).

We say that a function \(f : \Omega \rightarrow X\) respects an information partition for player \(i\) if \(f(\omega) = f(\omega')\) whenever \(\omega \in P_i\) and \(\omega' \in P_i\) for some \(P_i \in \mathcal{P}_i\).

Example: \(f\) respects \(\mathcal{P}_1\) if \(f(y) = f(z)\).
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Correlated Equilibria – Formally

**Definition**

A correlated equilibrium of a strategic game \( \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \) consists of

- a finite probability space \((\Omega, \pi)\),
- for each player \( i \in N \) an **information partition** \( \mathcal{P}_i \) of \( \Omega \),
- for each player \( i \in N \) a function \( \sigma_i : \Omega \rightarrow A_i \) that respects \( \mathcal{P}_i \) (**\( \sigma_i \) is player \( i \)'s strategy**)

such that for every \( i \in N \) and every function \( \tau_i : \Omega \rightarrow A_i \) that respects \( \mathcal{P}_i \) (**i.e. for every possible strategy of player \( i \)**) we have

\[
\sum_{\omega \in \Omega} \pi(\omega)u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega)u_i(\sigma_{-i}(\omega), \tau_i(\omega)). \tag{2}
\]
Example

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>6,6</td>
<td>2,7</td>
</tr>
<tr>
<td>B</td>
<td>7,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Equilibria: \((T, R)\) with \((2,7)\), \((B, L)\) with \((7,2)\), and mixed \(((\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}))\) with \((4\frac{2}{3}, 4\frac{2}{3})\).

Assume \(\Omega = \{x, y, z\}\), \(\pi(x) = \frac{1}{3}, \pi(y) = \frac{1}{3}, \pi(z) = \frac{1}{3}\).

Assume further \(P_1 = \{\{x\}, \{y, z\}\}\), \(P_2 = \{\{x, y\}, \{z\}\}\).

Set \(\sigma_1(x) = B, \sigma_1(y) = \sigma_1(z) = T\) and \(\sigma_2(x) = \sigma_2(y) = L, \sigma_2(z) = R\).

Then both player play optimally and get a payoff profile of \((5, 5)\).
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\begin{array}{cc}
L & R \\
T & 6,6 & 2,7 \\
B & 7,2 & 0,0 \\
\end{array}
\]

**Equilibria:** \((T, R)\) with \((2, 7)\), \((B, L)\) with \((7, 2)\), and mixed \(((\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}))\) with \((4\frac{2}{3}, 4\frac{2}{3})\).

Assume \(\Omega = \{x, y, z\}, \pi(x) = \frac{1}{3}, \pi(y) = \frac{1}{3}, \pi(z) = \frac{1}{3}\).
Assume further \(P_1 = \{\{x\}, \{y, z\}\}, P_2 = \{\{x, y\}, \{z\}\}\).
Set \(\sigma_1(x) = B, \sigma_1(y) = \sigma_1(z) = T\) and \(\sigma_2(x) = \sigma_2(y) = L, \sigma_2(z) = R\).

Then both player play optimally and get a payoff profile of \((5, 5)\).
Connection to Nash Equilibria

**Proposition**

For every mixed strategy Nash equilibrium $\alpha$ of a finite strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, there is a correlated equilibrium $\langle (\Omega, \pi), (P_i), (\sigma_i) \rangle$ in which for each player $i$ the distribution on $A_i$ induced by $\sigma_i$ is $\alpha_i$.

This means that correlated equilibria are a generalization of Nash equilibria.
Proof.

Let $\Omega = A$ and define $\pi(a) = \prod_{j \in N} \alpha_j(a_j)$. For each player $i$, let $a \in P$ and $b \in P$ for $P \in \mathcal{P}_i$ if $a_i = b_i$. Define $\sigma_i(a) = a_i$ for each $a \in A$.

Then $\langle (\Omega, \pi), (\mathcal{P}_i), (\sigma_i) \rangle$ is a correlated equilibrium since the left hand side of (2) is the Nash equilibrium payoff and for each player $i$ at least as good any other strategy $\tau_i$ respecting the information partition. Further, the distribution induced by $\sigma_i$ is $\alpha_i$. 
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$\square$
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Then \( \langle (\Omega, \pi), (\mathcal{P}_i), (\sigma_i) \rangle \) is a correlated equilibrium since the left hand side of (2) is the Nash equilibrium payoff and for each player \( i \) at least as good any other strategy \( \tau_i \) respecting the information partition. Further, the distribution induced by \( \sigma_i \) is \( \alpha_i \).
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□
Proposition

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. Any convex combination of correlated equilibrium payoff profiles of $G$ is a correlated equilibrium payoff profile of $G$.

Proof idea: From given equilibria and weighting factors, create a new one by combining them orthogonally, using the weighting factors.
Proof

Let $u^1, \ldots, u^K$ be the payoff profiles and let $(\lambda^1, \ldots, \lambda^K) \in \mathbb{R}^K$ with $\lambda^l \geq 0$ and $\sum_{l=1}^{K} \lambda^l = 1$. For each $l$ let $(\Omega^l, \pi^l), (\mathcal{P}_i^l), (\sigma_i^l)$ be a correlated equilibrium generating payoff $u^l$. Wlog. assume all $\Omega^l$'s are disjoint.

Now we define a correlated equilibrium generating the payoff $\sum_{l=1}^{K} \lambda^l u^l$. Let $\Omega = \bigcup_i \Omega^l$. For any $\omega \in \Omega$ define $\pi(\omega) = \lambda^l \pi^l(\omega)$ where $l$ is such that $\omega \in \Omega^l$. For each $i \in N$ let $\mathcal{P}_i = \bigcup_i \mathcal{P}_i^l$ and set $\sigma_i(\omega) = \sigma_i^l(\omega)$ where $l$ is such that $\omega \in \Omega^l$.

Basically, first throw a dice for which CE to go for, then proceed in this CE.
Proof.

Let $u_1, \ldots, u^K$ be the payoff profiles and let $(\lambda^1, \ldots, \lambda^K) \in \mathbb{R}^K$ with $\lambda^l \geq 0$ and $\sum_{l=1}^K \lambda^l = 1$. For each $l$ let $\langle (\Omega^l, \pi^l), (P_i^l), (\sigma_i^l) \rangle$ be a correlated equilibrium generating payoff $u^l$. Wlog. assume all $\Omega^l$’s are disjoint.

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Summary
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- **Mixed strategies** allow randomization.
- **Characterization** of mixed-strategy Nash equilibria: players only play best responses with positive probability (support lemma).
- **Nash’s Theorem**: Every finite strategic game has a mixed-strategy Nash equilibrium.
- **Correlated equilibria** can lead to higher payoffs.
- All Nash equilibria are correlated equilibria, but not *vice versa*. 