# Game Theory

## 3. Mixed Strategies

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# Mixed Strategies

# Mixed Strategies



Observation: Not every strategic game has a pure-strategy Nash equilibrium (e. g. matching pennies).

Question:

- Can we do anything about that?
- Which strategy to play then?

Idea: Consider randomized strategies.

## Notation

Let $X$ be a set.

Then $\Delta(X)$ denotes the set of probability distributions over $X$.

That is, each $p \in \Delta(X)$ is a mapping $p : X \to [0, 1]$ with

$$\sum_{x \in X} p(x) = 1.$$

# Mixed Strategies



A mixed strategy is a strategy where a player is allowed to randomize his action (throw a dice mentally and then act according to what he has decided to do for each outcome).

## Definition (Mixed strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A mixed strategy of player $i$ in $G$ is a probability distribution $\alpha_i \in \Delta(A_i)$ over player $i$'s actions.

For $a_i \in A_i$, $\alpha_i(a_i)$ is the probability for playing $a_i$.

Terminology: When we talk about strategies in $A_i$ specifically, to distinguish them from mixed strategies, we sometimes also call them pure strategies.

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## Definition (Mixed strategy profile)

A profile $\alpha = (\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$ of mixed strategies induces a probability distribution $p_\alpha$ over $A = \prod_{i \in N} A_i$ as follows:

$$p_\alpha(a) = \prod_{i \in N} \alpha_i(a_i).$$

For $A' \subseteq A$, we define

$$p_\alpha(A') = \sum_{a \in A'} p_\alpha(a) = \sum_{a \in A'} \prod_{i \in N} \alpha_i(a_i).$$

# Mixed Strategies



## Notation

Since each pure strategy $a_i \in A_i$ is equivalent to its induced mixed strategy $\hat{a}_i$

$$\hat{a}_i(a_i') = \begin{cases} 1 & \text{if } a_i' = a_i \\ 0 & \text{otherwise,} \end{cases}$$

we sometimes abuse notation and write $a_i$ instead of $\hat{a}_i$.

## Example (Mixed strategies for matching pennies)

| | $H$ | $T$ |
|---|---|---|
| $H$ | $1, -1$ | $-1, \ \ 1$ |
| $T$ | $-1, \ \ 1$ | $1, -1$ |

$\alpha = (\alpha_1, \alpha_2)$, $\alpha_1(H) = 2/3$, $\alpha_1(T) = 1/3$, $\alpha_2(H) = 1/3$, $\alpha_2(T) = 2/3$.

This induces a probability distribution over $\{H,T\} \times \{H,T\}$:

$$
\begin{aligned}
p_\alpha(H,H) &= \alpha_1(H) \cdot \alpha_2(H) = 2/9, & u_1(H,H) &= +1,\\
p_\alpha(H,T) &= \alpha_1(H) \cdot \alpha_2(T) = 4/9, & u_1(H,T) &= -1,\\
p_\alpha(T,H) &= \alpha_1(T) \cdot \alpha_2(H) = 1/9, & u_1(T,H) &= -1,\\
p_\alpha(T,T) &= \alpha_1(T) \cdot \alpha_2(T) = 2/9, & u_1(T,T) &= +1.
\end{aligned}
$$

# Expected Utility

## Definition (Expected utility)

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile.

The expected utility of $\alpha$ for player $i$ is

$$U_i(\alpha) = U_i\left((\alpha_j)_{j \in N}\right) := \sum_{a \in A} p_\alpha(a)\, u_i(a) = \sum_{a \in A} \left(\prod_{j \in N} \alpha_j(a_j)\right) u_i(a).$$

## Example (Mixed strategies for matching pennies (ctd.))

The expected utilities for player 1 and player 2 are

$$U_1(\alpha_1, \alpha_2) = -1/9 \qquad \text{and} \qquad U_2(\alpha_1, \alpha_2) = +1/9.$$

# Expected Utility

Remark: The expected utility functions $U_i$ are linear in all mixed strategies.

## Proposition

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile, $\beta_i, \gamma_i \in \Delta(A_i)$ mixed strategies, and $\lambda \in [0,1]$. Then

$$U_i(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i) = \lambda U_i(\alpha_{-i}, \beta_i) + (1-\lambda)U_i(\alpha_{-i}, \gamma_i).$$

Moreover,

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) \cdot U_i(\alpha_{-i}, a_i)$$

## Proof.

Homework. □

# Mixed Extension



## Definition (Mixed extension)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

The mixed extension of $G$ is the game $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$ where

- $\Delta(A_i)$ is the set of probability distributions over $A_i$ and
- $U_i : \prod_{j \in N} \Delta(A_j) \to \mathbb{R}$ assigns to each mixed strategy profile $\alpha$ the expected utility for player $i$ according to the induced probability distribution $p_\alpha$.

# Nash Equilibria in Mixed Strategies



**Definition (Nash equilibrium in mixed strategies)**

Let $G$ be a strategic game.

A Nash equilibrium in mixed strategies (or mixed-strategy Nash equilibrium) of $G$ is a Nash equilibrium in the mixed extension of $G$.

# Support



## Intuition:

- It does not make sense to assign positive probability to a pure strategy that is not a best response to what the other players do.
- Claim: A profile of mixed strategies $\alpha$ is a Nash equilibrium if and only if everyone only plays best pure responses to what the others play.

### Definition (Support)

Let $\alpha_i$ be a mixed strategy.

The support of $\alpha_i$ is the set

$$supp(\alpha_i) = \{a_i \in A_i \,|\, \alpha_i(a_i) > 0\}$$

of actions played with nonzero probability.

# Support Lemma



### Lemma (Support lemma)

*Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game.*

*Then $\alpha^* \in \prod_{i \in N} \Delta(A_i)$ is a mixed-strategy Nash equilibrium in $G$ if and only if for every player $i \in N$, every pure strategy in the support of $\alpha_i^*$ is a best response to $\alpha_{-i}^*$.*

For a single player–given all other players stick to their mixed strategies–it does not make a difference whether he plays the mixed strategy or whether he plays any single pure strategy from the support of the mixed strategy.

## Example (Support lemma)

Matching pennies, strategy profile $\alpha = (\alpha_1, \alpha_2)$ with

$$\alpha_1(H) = {}^2\!/_3, \quad \alpha_1(T) = {}^1\!/_3, \quad \alpha_2(H) = {}^1\!/_3, \text{ and } \quad \alpha_2(T) = {}^2\!/_3.$$

For $\alpha$ to be a Nash equilibrium, both actions in $supp(\alpha_2) = \{H, T\}$ have to be best responses to $\alpha_1$. Are they?

$$\begin{aligned} U_2(\alpha_1, H) &= \alpha_1(H) \cdot u_2(H, H) + \alpha_1(T) \cdot u_2(T, H) \\ &= {}^2\!/_3 \cdot (-1) + {}^1\!/_3 \cdot (+1) = -{}^1\!/_3, \\ U_2(\alpha_1, T) &= \alpha_1(H) \cdot u_2(H, T) + \alpha_1(T) \cdot u_2(T, T) \\ &= {}^2\!/_3 \cdot (+1) + {}^1\!/_3 \cdot (-1) = {}^1\!/_3. \end{aligned}$$

$\Rightarrow$ $H \in supp(\alpha_2)$, but $H \notin B_2(\alpha_1)$.

$\overset{\text{Support lemma}}{\Rightarrow}$ $\alpha$ can not be a Nash equilibrium.

# Support Lemma



**Proof.**

"$\Rightarrow$": Let $\alpha^*$ be a Nash equilibrium with $a_i \in supp(\alpha_i^*)$.

Assume that $a_i$ is not a best response to $\alpha_{-i}^*$. Because $U_i$ is linear, player $i$ can improve his utility by shifting probability in $\alpha_i^*$ from $a_i$ to a better response.

This makes the modified $\alpha_i^*$ a better response than the original $\alpha_i^*$, i. e., the original $\alpha_i^*$ was not a best response, which contradicts the assumption that $\alpha^*$ is a Nash equilibrium.

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# Support Lemma

## Proof (ctd.)

"$\Leftarrow$": Assume that $\alpha^*$ is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy $\alpha_i'$ such that $U_i(\alpha_{-i}^*, \alpha_i') > U_i(\alpha_{-i}^*, \alpha_i^*)$.

Because $U_i$ is linear, there must be a pure strategy $a_i' \in supp(\alpha_i')$ that has higher utility than some pure strategy $a_i'' \in supp(\alpha_i^*)$.

Therefore, $supp(\alpha_i^*)$ does not only contain best responses to $\alpha_{-i}^*$. □

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# Support Lemma



## Proof (ctd.)

"$\Leftarrow$": Assume that $\alpha^*$ is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy $\alpha_i'$ such that $U_i(\alpha^*_{-i}, \alpha_i') > U_i(\alpha^*_{-i}, \alpha_i^*)$.

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Therefore, $supp(\alpha_i^*)$ does not only contain best responses to $\alpha^*_{-i}$. □

# Computing Mixed-Strategy Nash Equilibria



## Example (Mixed-strategy Nash equilibria in BoS)

| | $B$ | $S$ |
|---|---|---|
| $B$ | 2,1 | 0,0 |
| $S$ | 0,0 | 1,2 |

We already know: $(B,B)$ and $(S,S)$ are pure Nash equilibria.

Possible supports (excluding "pure-vs-pure" strategies) are:

$\{B\}$ vs. $\{B,S\}$, $\{S\}$ vs. $\{B,S\}$, $\{B,S\}$ vs. $\{B\}$,
$\{B,S\}$ vs. $\{S\}$ and $\{B,S\}$ vs. $\{B,S\}$

Observation: In Bach or Stravinsky, pure strategies have unique best responses. Therefore, there can be no Nash equilibria of "pure-vs-strictly-mixed" type.

# Computing Mixed-Strategy Nash Equilibria



## Example (Mixed-strategy Nash equilibria in BoS (ctd.))

Consequence: Only need to search for additional Nash equilibria with support sets $\{B,S\}$ vs. $\{B,S\}$.

Assume that $(\alpha_1^*, \alpha_2^*)$ is a Nash equilibrium with $0 < \alpha_1^*(B) < 1$ and $0 < \alpha_2^*(B) < 1$. Then

$$\begin{aligned}
& U_1(B,\alpha_2^*) = U_1(S,\alpha_2^*) \\
\Rightarrow \quad & 2\cdot\alpha_2^*(B) + 0\cdot\alpha_2^*(S) = 0\cdot\alpha_2^*(B) + 1\cdot\alpha_2^*(S) \\
\Rightarrow \quad & 2\cdot\alpha_2^*(B) = 1 - \alpha_2^*(B) \\
\Rightarrow \quad & 3\cdot\alpha_2^*(B) = 1 \\
\Rightarrow \quad & \alpha_2^*(B) = 1/3 \quad (\text{and } \alpha_2^*(S) = 2/3)
\end{aligned}$$

Similarly, we get $\alpha_1^*(B) = 2/3$ and $\alpha_1^*(S) = 1/3$.

The payoff profile of this equilibrium is $(2/3, 2/3)$.

# Support Lemma

## Remark

Let $G = \langle \{1,2\}, (A_i), (u_i) \rangle$ with $A_1 = \{T,B\}$ and $A_2 = \{L,R\}$ be a two-player game with two actions each, and $(T, \alpha_2^*)$ with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of $G$.

Then at least one of the profiles $(T,L)$ and $(T,R)$ is also a Nash equilibrium of $G$.

Reason: Both $L$ and $R$ are best responses to $T$. Assume that $T$ was neither a best response to $L$ nor to $R$. Then $B$ would be a better response than $T$ both to $L$ and to $R$.

With the linearity of $U_1$, $B$ would also be a better response to $\alpha_2^*$ than $T$ is. Contradiction.

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With the linearity of $U_1$, $B$ would also be a better response to $\alpha_2^*$ than $T$ is. Contradiction.

# Support Lemma

## Example

Consider the Nash equilibrium $\alpha^* = (\alpha_1^*, \alpha_2^*)$ with

$$\alpha_1^*(T) = 1, \qquad \alpha_1^*(B) = 0, \qquad \alpha_2^*(L) = 1/10, \qquad \alpha_2^*(R) = 9/10$$

in the following game:

| | $L$ | $R$ |
|---|---|---|
| $T$ | 1, 1 | 1, 1 |
| $B$ | 2, 2 | −5, −5 |

Here, $(T, R)$ is also a Nash equilibrium.

# Nash's Theorem

# Nash's Theorem

Motivation: When does a strategic game have a mixed-strategy Nash equilibrium?

In the previous chapter, we discussed necessary and sufficient conditions for the existence of Nash equilibria for the special case of zero-sum games. Can we make other claims?

# Nash's Theorem



## Theorem (Nash's theorem)

*Every finite strategic game has a mixed-strategy Nash equilibrium.*

## Proof sketch.

Consider the set-valued function of best responses $B : \mathbb{R}^{\sum_i |A_i|} \to 2^{\mathbb{R}^{\sum_i |A_i|}}$ with

$$B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i}).$$

A mixed strategy profile $\alpha$ is a fixed point of $B$ if and only if $\alpha \in B(\alpha)$ if and only if $\alpha$ is a mixed-strategy Nash equilibrium.

The graph of $B$ has to be connected. Then there is at least one point on the fixpoint diagonal. $\square$

# Nash's Theorem



## Outline for the formal proof:

1. Review of necessary mathematical definitions
   $\rightsquigarrow$ Subsection "Definitions"
2. Statement of a fixpoint theorem used to prove Nash's theorem (without proof)
   $\rightsquigarrow$ Subsection "Kakutani's Fixpoint Theorem"
3. Proof of Nash's theorem using fixpoint theorem
   $\rightsquigarrow$ Subsection "Proof of Nash's Theorem"

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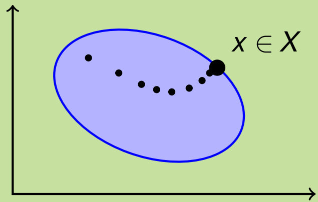
## Definition

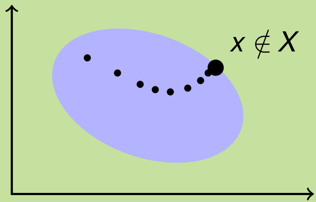
A set $X \subseteq \mathbb{R}^n$ is closed if $X$ contains all its limit points, i. e., if $(x_k)_{k\in\mathbb{N}}$ is a sequence of elements in $X$ and $\lim_{k\to\infty} x_k = x$, then also $x \in X$.

## Example

Closed:




Not closed:



# Nash's Theorem

## Definitions

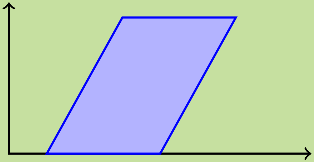
### Definition

A set $X \subseteq \mathbb{R}^n$ is bounded if for each $i = 1, \ldots, n$ there are lower and upper bounds $a_i, b_i \in \mathbb{R}$ such that

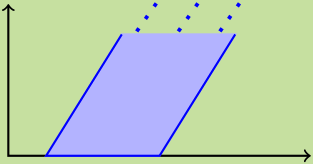
$$X \subseteq \prod_{i=1}^{n} [a_i, b_i].$$

### Example

Bounded:



Not bounded:



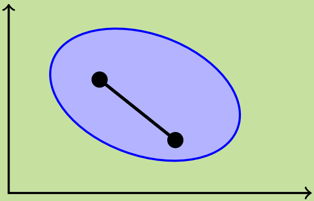
### Definition

A set $X \subseteq \mathbb{R}^n$ is convex if for all $x, y \in X$ and all $\lambda \in [0,1]$,

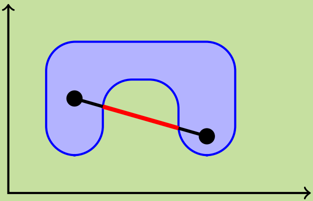
$$\lambda x + (1-\lambda) y \in X.$$

### Example

Convex:



Not convex:



## Definition

For a function $f : X \to 2^X$, the graph of $f$ is the set

$$Graph(f) = \{(x,y) \mid x \in X,\ y \in f(x)\}.$$

## Theorem (Kakutani's fixpoint theorem)

*Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed, bounded and convex set and let $f : X \to 2^X$ be a function such that*

- *for all $x \in X$, the set $f(x) \subseteq X$ is nonempty and convex, and*
- *$Graph(f)$ is closed.*

*Then there is an $x \in X$ with $x \in f(x)$, i. e., $f$ has a fixpoint.*

## Proof.

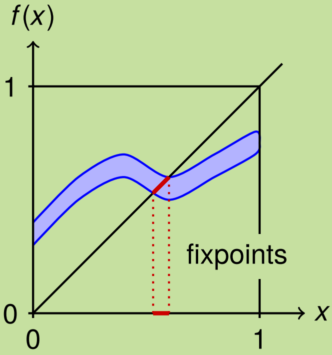
See Shizuo Kakutani, A generalization of Brouwer's fixed point theorem, 1941, or your favorite advanced calculus textbook, or the Internet.

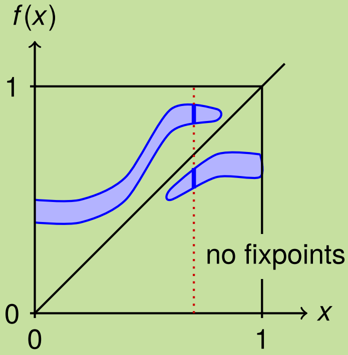
For German speakers: Harro Heuser, Lehrbuch der Analysis, Teil 2, also has a proof (Abschnitt 232). □

## Example

Let $X = [0,1]$.

Kakutani's theorem applicable: Kakutani's theorem not applicable:



## Proof.

Apply Kakutani's fixpoint theorem using $X = \mathscr{A} = \prod_{i \in N} \Delta(A_i)$ and $f = B$, where $B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i})$.

We have to show:

1. $\mathscr{A}$ is nonempty,
2. $\mathscr{A}$ is closed,
3. $\mathscr{A}$ is bounded,
4. $\mathscr{A}$ is convex,
5. $B(\alpha)$ is nonempty for all $\alpha \in \mathscr{A}$,
6. $B(\alpha)$ is convex for all $\alpha \in \mathscr{A}$, and
7. $Graph(B)$ is closed.

## Proof (ctd.)

### Some notation:

- Assume without loss of generality that $N = \{1, \dots, n\}$.
- A profile of mixed strategies can be written as a vector of $M = \sum_{i \in N} |A_i|$ real numbers in the interval $[0, 1]$ such that numbers for the same player add up to 1.

  For example, $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1(T) = 0.7$, $\alpha_1(M) = 0.0$, $\alpha_1(B) = 0.3$, $\alpha_2(L) = 0.4$, $\alpha_2(R) = 0.6$ can be seen as the vector

$$(\underbrace{0.7,\ 0.0,\ 0.3}_{\alpha_1},\ \underbrace{0.4,\ 0.6}_{\alpha_2})$$

- This allows us to interpret the set $\mathscr{A}$ of mixed strategy profiles as a subset of $\mathbb{R}^M$.

## Proof (ctd.)

1. $\mathscr{A}$ nonempty: Trivial. $\mathscr{A}$ contains the tuple

$$(1, \underbrace{0,\ldots,0}_{|A_1|-1 \text{ times}}, \ldots, 1, \underbrace{0,\ldots,0}_{|A_n|-1 \text{ times}}).$$

2. $\mathscr{A}$ closed: Let $\alpha_1, \alpha_2, \ldots$ be a sequence in $\mathscr{A}$ that converges to $\lim_{k\to\infty} \alpha_k = \alpha$. Suppose $\alpha \notin \mathscr{A}$. Then either there is some component of $\alpha$ that is less than zero or greater than one, or the components for some player $i$ add up to a value other than one.

   Since $\alpha$ is a limit point, the same must hold for some $\alpha_k$ in the sequence. But then, $\alpha_k \notin \mathscr{A}$, a contradiction. Hence $\mathscr{A}$ is closed.

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## Proof (ctd.)

3. $\mathscr{A}$ bounded: Trivial. All entries are between 0 and 1, i. e., $\mathscr{A}$ is bounded by $[0,1]^M$.

4. $\mathscr{A}$ convex: Let $\alpha, \beta \in \mathscr{A}$ and $\lambda \in [0,1]$, and consider $\gamma = \lambda\alpha + (1-\lambda)\beta$. Then

$$\begin{aligned}\min(\gamma) &= \min(\lambda\alpha + (1-\lambda)\beta) \\ &\geq \lambda \cdot \min(\alpha) + (1-\lambda) \cdot \min(\beta) \\ &\geq \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0,\end{aligned}$$

and similarly, $\max(\gamma) \leq 1$.

Hence, all entries in $\gamma$ are still in $[0,1]$.

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$$\begin{aligned}\min(\gamma) &= \min(\lambda\alpha + (1-\lambda)\beta) \\ &\geq \lambda \cdot \min(\alpha) + (1-\lambda) \cdot \min(\beta) \\ &\geq \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0,\end{aligned}$$

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Hence, $U_i$ is continous on $\Delta(A_i)$.

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## Proof (ctd.)

7 *Graph*(*B*) closed (ctd.): It holds for all $i \in N$:

$$
\begin{aligned}
U_i\left(\alpha_{-i}, \beta_i\right) &\overset{\text{(D)}}{=} U_i\left(\lim_{k\to\infty}(\alpha^k_{-i}, \beta^k_i)\right) \\
&\overset{\text{(C)}}{=} \lim_{k\to\infty} U_i\left(\alpha^k_{-i}, \beta^k_i\right) \\
&\overset{\text{(B)}}{\geq} \lim_{k\to\infty} U_i\left(\alpha^k_{-i}, \beta'_i\right) \qquad \text{for all } \beta'_i \in \Delta(A_i) \\
&\overset{\text{(C)}}{=} U_i\left(\lim_{k\to\infty} \alpha^k_{-i}, \beta'_i\right) \qquad \text{for all } \beta'_i \in \Delta(A_i) \\
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(D): def. $\alpha_i$, $\beta_i$; (C) continuity; (B) $\beta^k_i$ best response to $\alpha^k_{-i}$.

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7. *Graph*($B$) closed (ctd.): It follows that $\beta_i$ is a best response to $\alpha_{-i}$ for all $i \in N$.

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Applying Kakutani's theorem establishes the existence of a fixpoint of $B$, which is, by definition/construction, the same as a mixed-strategy Nash equilibrium. □

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# Correlated Equilibria

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Recall: There are three Nash equilibria in Bach or Stravinsky

- $(B,B)$ with payoff profile $(2,1)$
- $(S,S)$ with payoff profile $(1,2)$
- $(\alpha_1^*, \alpha_2^*)$ with payoff profile $(2/3, 2/3)$ where
  - $\alpha_1^*(B) = 2/3$, $\alpha_1^*(S) = 1/3$,
  - $\alpha_2^*(B) = 1/3$, $\alpha_2^*(S) = 2/3$.

Idea: Use a publicly visible coin toss to decide which action from a mixed strategy is played. This can lead to higher payoffs.

# Correlated Equilibria



## Example (Correlated equilibrium in BoS)

With a fair coin that both players can observe, the players can agree to play as follows:

- If the coin shows heads, both play $B$.
- If the coin shows tails, both play $S$.

This is stable in the sense that no player has an incentive to deviate from this agreed-upon rule, as long as the other player keeps playing his/her strategy (cf. definition of Nash equilibria).

Expected payoffs: $(3/2, 3/2)$ instead of $(2/3, 2/3)$.

# Observations and Information Partitions

We assume that observations are made based on a finite probability space $(\Omega, \pi)$, where $\Omega$ is a set of states and $\pi$ is a probability measure on $\Omega$.

Agents might not be able to distingush all states from each other. In order to model this, we assume for each player $i$ an information partition $\mathscr{P}_i = \{P_{i1}, P_{i2}, \dots, P_{ik}\}$. This means that $\bigcup_{j=1}^{j=ik} P_j = \Omega$ and for all $P_i, P_k \in \mathscr{P}_i$ with $P_j \neq P_k$, we have $P_j \cap P_k = \emptyset$.

Example: $\Omega = \{x, y, z\}$, $\mathscr{P}_1 = \{\{x\}, \{y, z\}\}$, $\mathscr{P}_2 = \{\{x, y\}, \{z\}\}$.

We say that a function $f : \Omega \to X$ respects an information partition for player $i$ if $f(\omega) = f(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ for some $P_i \in \mathscr{P}_i$.

Example: $f$ respects $\mathscr{P}_1$ if $f(y) = f(z)$.

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# Correlated Equilibria – Formally



## Definition

A correlated equilibrium of a strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ consists of

- a finite probability space $(\Omega, \pi)$,
- for each player $i \in N$ an information partition $\mathscr{P}_i$ of $\Omega$,
- for each player $i \in N$ a function $\sigma_i : \Omega \to A_i$ that respects $\mathscr{P}_i$ ($\sigma_i$ is player $i$'s strategy)

such that for every $i \in N$ and every function $\tau_i : \Omega \to A_i$ that respects $\mathscr{P}_i$ (i.e. for every possible strategy of player $i$) we have

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)). \qquad (2)$$

# Example



| | $L$ | $R$ |
|---|---|---|
| $T$ | $6,6$ | $2,7$ |
| $B$ | $7,2$ | $0,0$ |

Equilibria: $(T,R)$ with $(2,7)$, $(B,L)$ with $(7,2)$, and mixed $((\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}))$ with $(4\frac{2}{3},4\frac{2}{3})$.

Assume $\Omega = \{x,y,z\}, \pi(x) = \frac{1}{3}, \pi(y) = \frac{1}{3}, \pi(z) = \frac{1}{3}$.
Assume further $\mathscr{P}_1 = \{\{x\},\{y,z\}\}, \mathscr{P}_2 = \{\{x,y\},\{z\}\}$.
Set $\sigma_1(x) = B, \sigma_1(y) = \sigma_1(z) = T$ and $\sigma_2(x) = \sigma_2(y) = L, \sigma_2(z) = R$.

Then both player play optimally and get a payoff profile of $(5,5)$.

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Set $\sigma_1(x) = B, \sigma_1(y) = \sigma_1(z) = T$ and $\sigma_2(x) = \sigma_2(y) = L, \sigma_2(z) = R$.

Then both player play optimally and get a payoff profile of $(5,5)$.

# Example



| | $L$ | $R$ |
|---|---|---|
| $T$ | $6,6$ | $2,7$ |
| $B$ | $7,2$ | $0,0$ |

Equilibria: $(T,R)$ with $(2,7)$, $(B,L)$ with $(7,2)$, and mixed $((\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}))$ with $(4\frac{2}{3},4\frac{2}{3})$.

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# Connection to Nash Equilibria



## Proposition

For every mixed strategy Nash equilibrium $\alpha$ of a finite strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, there is a correlated equilibrium $\langle (\Omega, \pi), (\mathscr{P}_i), (\sigma_i) \rangle$ in which for each player $i$ the distribution on $A_i$ induced by $\sigma_i$ is $\alpha_i$.

This means that correlated equilibria are a generalization of Nash equilibria.

# Proof

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Mixed Strategies
Nash's Theorem
Correlated Equilibria
Summary

**Proof.**

Let $\Omega = A$ and define $\pi(a) = \prod_{j \in N} \alpha_j(a_j)$. For each player $i$, let $a \in P$ and $b \in P$ for $P \in \mathscr{P}_i$ if $a_i = b_i$. Define $\sigma_i(a) = a_i$ for each $a \in A$.

Then $\langle(\Omega, \pi), (\mathscr{P}_i), (\sigma_i)\rangle$ is a correlated equilibrium since the left hand side of (2) is the Nash equilibrium payoff and for each player $i$ at least as good any other strategy $\tau_i$ respecting the information partition. Further, the distribution induced by $\sigma_i$ is $\alpha_i$. □

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**Proposition**

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. Any convex combination of correlated equlibirum payoff profiles of $G$ is a correlated equilibirum payoff profile of G.

Proof idea: From given equilibria and weighting factors, create a new one by combining them orthogonally, using the weighting factors.

**Proof.**

Let $u^1, \ldots, u^K$ be the payoff profiles and let $(\lambda^1, \ldots, \lambda^K) \in \mathbb{R}^K$ with $\lambda^l \geq 0$ and $\sum_{l=1}^{K} \lambda^l = 1$. For each $l$ let $\langle (\Omega^l, \pi^l), (\mathscr{P}_i^l), (\sigma_i^l) \rangle$ be a correlated equilibrium generating payoff $u^l$. Wlog. assume all $\Omega^l$'s are disjoint.

Now we define a correlated equilibrium generating the payoff $\sum_{l=1}^{K} \lambda^l u^l$. Let $\Omega = \bigcup_l \Omega^l$. For any $\omega \in \Omega$ define $\pi(\omega) = \lambda^l \pi^l(\omega)$ where $l$ is such that $\omega \in \Omega^l$. For each $i \in N$ let $\mathscr{P}_i = \bigcup_l \mathscr{P}_i^l$ and set $\sigma_i(\omega) = \sigma_i^l(\omega)$ where $l$ is such that $\omega \in \Omega^l$. $\square$

Basically, first throw a dice for which CE to go for, then proceed in this CE.

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# Summary

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- Mixed strategies allow randomization.
- Characterization of mixed-strategy Nash equilibria: players only play best responses with positive probability (support lemma).
- Nash's Theorem: Every finite strategic game has a mixed-strategy Nash equilibrium.
- Correlated equilibria can lead to higher payoffs.
- All Nash equilibria are correlated equilibria, but not *vice versa*.