10. Repeated Games

If a strategic game is played repeatedly, then the players might behave differently than in the one-shot setting.

a) finitely repeated: you play the game for a known number of rounds.

b) infinitely repeated: infinite number of rounds.

c) indefinitely repeated: you have a given probability $p$ that the current round is the last one.
Def.: Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game (the "stage game"). Let $A = \prod_{i=1}^{n} A_i$. Then a repeated game with $K \in \mathbb{N} \cup \{0\}$ moves is an extensive game with simultaneous moves $\Gamma = \langle \mathcal{N}, A, H, P, (\nu_i) \rangle$ with

- $H = \{()\} \cup \bigcup_{t=1}^{K} A^t$
- $P(h) = N$ for all nonzero hist. $h$
- If $k \in \mathbb{N}$: $\nu_i(h) = \sum_{t=1}^{k} u_i(a_t)$ for $h = a_1^t a_2^t \ldots a_k^t$ (for turn. hist. $h$)
If \( k = \infty \): Let \( \delta \in (0, 1) \) be a discount factor. Then \( v_i(h) = \sum_{t=1}^{\infty} \delta^{t-1} \cdot u_i(a_t) \)

\[ h = a_1 a_2 \ldots a_t \ldots \]

Example: \( \delta = \frac{1}{2} \). \( u_i(a_t) = 1 \)

\[ v_i(h) = \sum_{t=1}^{\infty} \delta^{t-1} \cdot u_i(a_t) = 1 \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 1 + \ldots \]

\[ = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2 \]
Example: Finitely repeated PD with \( k = 2 \).

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>3,3</td>
<td>0,4</td>
</tr>
<tr>
<td>D</td>
<td>4,0</td>
<td>1,1</td>
</tr>
</tbody>
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- No matter how many rounds \( k \in \mathbb{N} \) are played, DD is always the unique subgame-perfect equilibrium.
Infinitely repeated games

Discounting: What is \( 1 + \delta + \delta^2 + \delta^3 + \ldots \)?

It converges to \( \frac{1}{1-\delta} \) for \( 0 < \delta < 1 \).

Proof: \( x = 1 + \delta + \delta^2 + \delta^3 + \ldots \)

\[ = 1 + \delta x \]

\( \Rightarrow x - \delta x = 1 \)

\( \Rightarrow x(1-\delta) = 1 \)

\( \Rightarrow x = \frac{1}{1-\delta} \)
Strategies for infinitely repeated games:

Finite automata (Moore automata):

Example:

\[ P_1 : \text{C} \xrightarrow{D} P_2 : \text{D} \]

Grim strategy, \( g \)

Defecting strategy, \( d \)
Is \((g,g)\) an equilibrium?

\[ V^\alpha(g,g) = 3 + 8 \cdot 3 + 8^2 \cdot 3 + \ldots = 3 \cdot \frac{1}{1-8} = 3 \cdot \frac{1}{1-8} \]

Then, the unique run the players get is \((C,C), (C,C), (C,C), \ldots\).
Is \((g, g)\) a NE equilibrium?

Only reasonable candidate(s) for belief responses to \(g\):

\(g'\) : choose D at some point \textit{unprovoked},

and then \textit{ever after}

\[ \Rightarrow \ O(g', g) = \langle (C, C), (C, C), \ldots, (C, C),
\]

\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \end{array} \]

W.L.O.G., assume that the first \textit{unprovoked} Defection in \(g'\)

\(\text{happens in the first step.}\)

\[ \Rightarrow \ g' = d. \]
So: Determine $v_4(\mathbf{d}, g)$, compare to $v_4(\mathbf{d}, g) = \frac{3}{1-\delta}$.

We already know $v_4(\mathbf{d}, g) = 3 \cdot \frac{1}{1-\delta}$.

$v_4(\mathbf{d}(\mathbf{d}, g)) = v_4(\langle (D, C), (D, D), (D, D), \ldots \rangle)$

$= 4 + 8 \cdot 1 + 8^2 \cdot 1 + \ldots$

$= 4 + 8 \cdot \left(1 + 8 + 8^2 + \ldots \right)$

$= 4 + 8 \cdot \frac{1}{1-8} = 4 + \frac{8}{1-8}$

$(g, g)$ is a NE if $3 \cdot \frac{1}{1-\delta} \geq 4 + \frac{8}{1-8}$

$\Rightarrow 3 \geq 4 \left(1-\delta\right) + \delta = 4 - 4\delta + \delta = 4 - 3\delta$

$\Rightarrow 3\delta \geq 1 \Rightarrow \delta \geq \frac{1}{3}$
This means: \((g,g)\) is a NE if \(S \geq \frac{1}{3}\) is large enough.
(then, \(g\) is at least as good a response to \(g\) as \(d\) is.)

- Also, \((d,d)\) is a NE!

- Also, \((t,t)\) is a NE (for \(S > \frac{1}{2}\)).

Positive message: In repeated games, there are other NEs than just \((D,D)\).

Negative message: Which NE to play?
Indefinitely repeated games:

Theory very similar to theory for infinitely repeated games, because:

discount factor is probability that there is a next round.