10. Repeated Games

- If a strategic game is played repeatedly, then
  the players might behave differently than in the one-shot
  setting.

  a) finitely repeated: you play the game for a known
     number of rounds.
  b) infinitely repeated: infinite number of rounds.
  c) indefinitely repeated: you have a given probability \( p \) that
     the current round is the last one.

For \( k = \infty \): Let \( \delta \in (0, 1) \) be a discount factor.
Then \( v_i(\mathbf{w}) = \sum_{t=1}^{\infty} \delta^{t-1} u_i(a_t) \)

\[ \mathbf{w} = a^t a^{t-1} \ldots \]

Example: \( \delta = \frac{1}{2} \). \( u_i(a^t) = 1 \)

\[ v_i(\mathbf{w}) = \sum_{t=1}^{\infty} \delta^{t-1} u_i(a_t) = 1 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{2} + \ldots = 2 \]

Example: Finitely repeated PD with \( k = 2 \).

<table>
<thead>
<tr>
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<th>C</th>
<th>D</th>
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<tbody>
<tr>
<td>C</td>
<td>3, 3</td>
<td>0, 4</td>
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<tr>
<td>D</td>
<td>4, 0</td>
<td>1, 1</td>
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Example: Perfectly repeated PD with \( k = 2 \).

- No matter how many rounds \( k \) are played, DD is always the unique
  subgame perfect equilibrium.
**Definitely repeated games**

**Discounting:** What is \( A + S + S^2 + S^3 + \ldots \) ?

It converges to \( \frac{A}{1 - S} \) for \( 0 < S < 1 \).

**Proof:**

\[
\begin{align*}
X &= A + S + S^2 + S^3 + \ldots \\
&= A + S \left( X + S + S^2 + \ldots \right) \\
&= A + SX \\
\Rightarrow X - SX &= A \\
\Rightarrow X(1 - S) &= A \\
\Rightarrow X &= \frac{A}{1 - S}
\end{align*}
\]

**Strategies for infinitely repeated games:**

**Finite automata (Moore automata):**

**Example:**

\[ \xrightarrow{C} P_a : C \xrightarrow{D} P_a : D \]

- Grim strategy, \( g \)
- Defection strategy, \( d \)

**Is \((g,d)\) a N.E. equilibrium?**

Only one candidate for better response to \( g \):

- \( g' \): choose \( D \) at some point unpunished

And then never again:

\[ \Delta(g', g) = \{(C, C), (C, C), \ldots, (C, C), (D, C), (D, D), \ldots, (D, D), \ldots\} \]

U.C.O.G., assume that the first unpunished defection in \( g' \) happens in the first step:

\[ g' = d \]
So: Define $v(\{a, b\})$, compare to $v(\{a, g\})$.
We already know $v(\{a, g\}) = \frac{3 \cdot A}{1 - \delta}$.

$v_4(0, \{a, g\}) = v_4((\{b, c\}, \{b, d\}, \{b, d\}, \ldots))$

$= 4 + 8 \cdot \frac{A}{1 - \delta} + \frac{8^2 \cdot A}{1 - \delta} + \ldots$

$= 4 + 8 \cdot \frac{A}{1 - \delta} \cdot \frac{1}{1 - \delta} = 4 + \frac{8 \cdot A}{1 - \delta}$

$(g, g)$ is a NE if $\frac{3 \cdot \frac{A}{1 - \delta}}{1 - \delta} \geq 4 + \frac{8 \cdot A}{1 - \delta}$

$\Leftrightarrow 3 \cdot \frac{A}{1 - \delta} \geq 4 + 8 \cdot \frac{A}{1 - \delta}$

$\Leftrightarrow 3 \cdot A \geq 4 \cdot (1 - \delta) + 8 \cdot A \geq 4 - 4 \cdot \delta + 8 \cdot \delta = 4 - 3 \cdot \delta$

$\Leftrightarrow 3 \cdot \delta \geq 4 \Leftrightarrow \delta \geq \frac{4}{3}$

This means: $(g, g)$ is a NE if $\delta \geq \frac{4}{3}$.
(Then, $g$ is at least as good a response to $g$ as $d$ is.)

Also, $(d, d)$ is a NE!

Also, $(b, b)$ is a NE (for $\delta \geq \frac{4}{3}$).

Positive message: In repeated games, then are other NEs
than just $(b, b)$.

Negative message: What NE to play?

Indefinitely repeated games:

Theory very similar to theory for definitely repeated
games, because:

- Discount factor $\delta$ probability that there is a next round.