Preference relations $\prec$ contain no information about “by how much” one candidate is preferred.

**Idea:** Use money to measure this.

Use money also for transfers between players “for compensation”.

Formalization:

- Set of alternatives $A$.
- Set of $n$ players $I$.
- Valuation functions $v_i : A \rightarrow \mathbb{R}$ such that $v_i(a)$ denotes the value player $i$ assigns to alternative $a$.
- Payment functions specifying amount $p_i \in \mathbb{R}$ that player $i$ pays.
- Utility of player $i$: $u_i(a) = v_i(a) - p_i$. 
Second Price Auctions
Second price auctions:

- There are $n$ players bidding for a single item.
- Player $i$’s private valuations of item: $w_i$.
- Desired outcome: Player with highest private valuation wins bid.
- Players should reveal their valuations truthfully.
- Winner $i$ pays price $p^*$ and has utility $w_i - p^*$.
- Non-winners pay nothing and have utility 0.
Second Price Auctions

Formally:

- $A = N$
- $v_i(a) = \begin{cases} w_i & \text{if } a = i \\ 0 & \text{else} \end{cases}$

What about payments? Say player $i$ wins:

- $p^* = 0$ (winner pays nothing): bad idea, players would manipulate and publicly declare values $w'_i \gg w_i$.
- $p^* = w_i$ (winner pays his valuation): bad idea, players would manipulate and publicly declare values $w'_i = w_i - \varepsilon$.
- better: $p^* = \max_{j \neq i} w_j$ (winner pays second highest bid).
Vickrey Auction

**Definition (Vickrey Auction)**

The winner of the **Vickrey Auction** (aka second price auction) is the player \( i \) with the highest declared value \( w_i \). He has to pay the second highest declared bid \( p^* = \max_{j \neq i} w_j \).

**Proposition (Vickrey)**

Let \( i \) be one of the players and \( w_i \) his valuation for the item, \( u_i \) his utility if he truthfully declares \( w_i \) as his valuation of the item, and \( u'_i \) his utility if he falsely declares \( w'_i \) as his valuation of the item. Then \( u_i \geq u'_i \).

**Proof**

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Proof

See 

Incentive Compatible Mechanisms
**Idea:** Generalization of Vickrey auctions.

Preferences modeled as functions $v_i : A \rightarrow \mathbb{R}$.

Let $V_i$ be the space of all such functions for player $i$.

Unlike for social choice functions: Not only decide about chosen alternative, but also about payments.
Mechanisms

Definition (Mechanism)

A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) consists of

- a social choice function \( f: V_1 \times \cdots \times V_n \to A \) and
- for each player \( i \), a payment function \( p_i: V_1 \times \cdots \times V_n \to \mathbb{R} \).

Definition (Incentive Compatibility)

A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) is called incentive compatible if for each player \( i = 1, \ldots, n \), for all preferences \( v_1 \in V_1, \ldots, v_n \in V_n \) and for each preference \( v'_i \in V_i \),

\[
v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}).
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VCG Mechanisms
VCG Mechanisms

- If \( \langle f, p_1, \ldots, p_n \rangle \) is incentive compatible, truthfully declaring one's preference is dominant strategy.

- The Vickrey-Clarke-Groves mechanism is an incentive compatible mechanism that maximizes “social welfare”, i.e., the sum over all individual utilities \( \sum_{i=1}^n v_i(a) \).

- **Idea**: Reflect other players’ utilities in payment functions, align all players’ incentives with goal of maximizing social welfare.
VCG Mechanisms

Definition (Vickrey-Clarke-Groves mechanism)

A mechanism $\langle f, p_1, \ldots, p_n \rangle$ is called a Vickrey-Clarke-Groves mechanism (VCG mechanism) if

1. $f(v_1, \ldots, v_n) \in \arg\max_{a \in A} \sum_{i=1}^{n} v_i(a)$ for all $v_1, \ldots, v_n$ and
2. there are functions $h_1, \ldots, h_n$ with $h_i : V_{-i} \rightarrow \mathbb{R}$ such that $p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \ldots, v_n))$ for all $i = 1, \ldots, n$ and $v_1, \ldots, v_n$.

Note: $h_i(v_{-i})$ independent of player $i$’s declared preference $\Rightarrow h_i(v_{-i}) = c$ constant from player $i$’s perspective.

Utility of player $i = v_i(f(v_1, \ldots, v_n)) + \sum_{j \neq i} v_j(f(v_1, \ldots, v_n)) - c = \sum_{j=1}^{n} v_j(f(v_1, \ldots, v_n)) - c = \text{social welfare} - c$. 
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Theorem (Vickrey-Clarke-Groves)
Every VCG mechanism is incentive compatible.

Proof
Let \( i, v_{-i}, v_i \) and \( v_i' \) be given. Show: Declaring true preference \( v_i \) dominates declaring false preference \( v_i' \).

Let \( a = f(v_i, v_{-i}) \) and \( a' = f(v_i', v_{-i}) \).

Utility player \( i = \begin{cases} v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i}) & \text{if declaring } v_i \\ v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) & \text{if declaring } v_i' \end{cases} \)

Alternative \( a = f(v_i, v_{-i}) \) maximizes social welfare
\[ \Rightarrow v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a'). \]
\[ \Rightarrow v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i}). \]
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Clarke Pivot Rule

- **So far:** payment functions $p_i$ and functions $h_i$ unspecified.

- **One possibility:** $h_i(v_{-i}) = 0$ for all $h_i$ and $v_{-i}$.
  - **Drawback:** Too much money distributed among players (more that necessary).

- **Further requirements:**
  - Players should pay at most as much as they value the outcome.
  - Players should only pay, never receive money.
**Definition (individual rationality)**

A mechanism is **individually rational** if all players always get a nonnegative utility, i.e., if for all $i = 1, \ldots, n$ and all $v_1, \ldots, v_n$,

$$v_i(f(v_1, \ldots, v_n)) - p_i(v_1, \ldots, v_n) \geq 0.$$ 

**Definition (positive transfers)**

A mechanism has **no positive transfers** if no player is ever paid money, i.e., for all preferences $v_1, \ldots, v_n$,

$$p_i(v_1, \ldots, v_n) \geq 0.$$
Clarke Pivot Function

Definition (Clarke pivot function)

The **Clarke pivot function** is the function

\[ h_i(v_{-i}) = \max_{b \in A} \sum_{j \neq i} v_j(b). \]

- This leads to payment functions
  
  \[ p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \]

  for \( a = f(v_1, \ldots, v_n) \).

- Player \( i \) pays the difference between what the other players could achieve without him and what they achieve with him.

- Each player internalizes the externalities he causes.
Example

- **Players** $I = \{1, 2\}$, **alternatives** $A = \{a, b\}$.
- **Values**: $v_1(a) = 10$, $v_1(b) = 2$, $v_2(a) = 9$ and $v_2(b) = 15$.
- Without player 1: $b$ best, since $v_2(b) = 15 > 9 = v_2(a)$.
- With player 1: $a$ best, since $v_1(a) + v_2(a) = 10 + 9 = 19 > 17 = 2 + 15 = v_1(b) + v_2(b)$.
- With player 1, other players (i.e., player 2) lose $v_2(b) - v_2(a) = 6$ units of utility.

$\Rightarrow$ **Clarke pivot function** $h_1(v_2) = 15$

$\Rightarrow$ **payment function**

$$p_1(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq 1} v_j(b) - \sum_{j \neq 1} v_j(a) = 15 - 9 = 6.$$
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Clarke Pivot Rule

**Lemma (Clarke pivot rule)**

A VCG mechanism with Clarke pivot functions has no positive transfers. If $v_i(a) \geq 0$ for all $i = 1, \ldots, n$, $v_i \in V_i$ and $a \in A$, then the mechanism is also individually rational.

**Proof**

Let $a = f(v_1, \ldots, v_n)$ be the alternative maximizing $\sum_{j=1}^{n} v_j(a)$, and $b$ the alternative maximizing $\sum_{j \neq i} v_j(b)$.

Utility of player $i$: $u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b)$.

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Since $b$ maximizes $\sum_{j \neq i} v_j(b)$: $p_i(v_1, \ldots, v_n) \geq 0$ (no positive transfers).
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(no positive transfers).
Clarke Pivot Rule

Proof (ctd.)

Individual rationality: Since $v_i(b) \geq 0$, 

$$u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_{j=1}^{n} v_j(a) - \sum_{j=1}^{n} v_j(b).$$

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Therefore, the mechanism is also individually rational.
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Therefore, the mechanism is also individually rational.
Vickrey Auction as a VCG Mechanism

- $A = N$. Valuations: $w_i, v_a(a) = w_a, v_i(a) = 0$ ($i \neq a$).
- $a$ maximizes social welfare $\sum_{i=1}^{n} v_i(a)$ iff $a$ maximizes $w_a$.
- Let $a = f(v_1, \ldots, v_n) = \arg \max_{j \in A} w_j$ be the highest bidder.
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\]
- Non-winners pay nothing: For $i \neq a$,
  \[
p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)
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\]
Vickrey Auction as a VCG Mechanism

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  $$= \max_{b \in A \setminus \{i\}} w_b - w_a = w_a - w_a = 0.$$
Example: Bilateral Trade

- **Seller** $s$ offers item he values with $0 \leq w_s \leq 1$.
- Potential **buyer** $b$ values item with $0 \leq w_b \leq 1$.
- Alternatives $A = \{\text{trade}, \text{no-trade}\}$.
- Valuations:
  \[
  \begin{align*}
  v_s(\text{no-trade}) &= 0, & v_s(\text{trade}) &= -w_s, \\
  v_b(\text{no-trade}) &= 0, & v_b(\text{trade}) &= w_b.
  \end{align*}
  \]
- VCG mechanism maximizes $v_s(a) + v_b(a)$.
- We have
  \[
  \begin{align*}
  v_s(\text{trade}) + v_b(\text{trade}) &= w_b - w_s, \\
  v_s(\text{no-trade}) + v_b(\text{no-trade}) &= 0
  \end{align*}
  \]
  i.e., $\text{trade}$ maximizes social welfare iff $w_b \geq w_s$. 
**Example: Bilateral Trade (ctd.)**

- **Requirement:** if *no-trade* is chosen, neither player pays anything:
  \[ p_s(v_s, v_b) = p_b(v_s, v_b) = 0. \]

- **To that end,** choose Clarke pivot function for **buyer**:
  \[ h_b(v_s) = \max_{a \in A} v_s(a). \]

- **For seller:** Modify Clarke pivot function by an additive constant and set
  \[ h_s(v_b) = \max_{a \in A} v_b(a) - w_b. \]
Example: Bilateral Trade (ctd.)

- For alternative *no-trade*,

\[ p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\textit{no-trade}) \]

\[ = w_b - w_b - 0 = 0 \quad \text{and} \]

\[ p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\textit{no-trade}) \]

\[ = 0 - 0 = 0. \]

- For alternative *trade*,

\[ p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\textit{trade}) \]

\[ = w_b - w_b - w_b = -w_b \quad \text{and} \]

\[ p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\textit{trade}) \]

\[ = 0 + w_s = w_s. \]
Example: Bilateral Trade (ctd.)

- Because $w_b \geq w_s$, the seller gets at least as much as the buyer pays, i.e., the mechanism subsidizes the trade.
- Without subsidies, no incentive compatible bilateral trade possible.
- Note: Buyer and seller can exploit the system by colluding.
Example: Public Project

- Project costs $C$ units.
- Each citizen $i$ privately values the project at $w_i$ units.
- Government will undertake project if $\sum_i w_i > C$.
- Alternatives: $A = \{\text{project, no-project}\}$.
- Valuations:

  \[
  v_G(\text{project}) = -C, \quad v_G(\text{no-project}) = 0,
  
  v_i(\text{project}) = w_i, \quad v_i(\text{no-project}) = 0.
  \]

- VCG mechanism with Clarke pivot rule: for each citizen $i$,

  \[
  h_i(v_{-i}) = \max_{a \in A} \left( \sum_{j \neq i} v_j(a) + v_G(a) \right)
  \]

  \[
  = \begin{cases} 
  \sum_{j \neq i} w_j - C, & \text{if } \sum_{j \neq i} w_j > C \\
  0, & \text{otherwise}.
  \end{cases}
  \]
Example: Public Project (ctd.)

- Citizen \( i \) pivotal if \( \sum_j w_j > C \) and \( \sum_{j \neq i} w_j \leq C \).

- Payment function for citizen \( i \):

\[
p_i(v_{1..n}, v_G) = h_i(v_{-i}) - \left( \sum_{j \neq i} v_j(f(v_{1..n}, v_G)) + v_G(f(v_{1..n}, v_G)) \right)
\]

- Case 1: Project undertaken, \( i \) pivotal:

\[
p_i(v_{1..n}, v_G) = 0 - \left( \sum_{j \neq i} w_j - C \right) = C - \sum_{j \neq i} w_j
\]

- Case 2: Project undertaken, \( i \) not pivotal:

\[
p_i(v_{1..n}, v_G) = \left( \sum_{j \neq i} w_j - C \right) - \left( \sum_{j \neq i} w_j - C \right) = 0
\]

- Case 3: Project not undertaken:

\[
p_i(v_{1..n}, v_G) = 0
\]
Example: Public Project (ctd.)

- I.e., citizen \( i \) pays nonzero amount

\[
C - \sum_{j \neq i} w_j
\]

only if he is pivotal.

- He pays difference between value of project to fellow citizens and cost \( C \), in general less than \( w_i \).

- Generally,

\[
\sum_i p_i(\text{project}) \leq C
\]

i.e., project has to be subsidized.
Example: Buying a Path in a Network

- Communication network modeled as $G = (V, E)$.
- Each link $e \in E$ owned by different player $e$.
- Each link $e \in E$ has cost $c_e$ if used.
- **Objective:** procure communication path from $s$ to $t$.
- **Alternatives:** $A = \{p \mid p$ path from $s$ to $t\}$.
- **Valuations:** $v_e(p) = -c_e$, if $e \in p$, and $v_e(p) = 0$, if $e \notin p$.
- **Maximizing social welfare:**
  
  
  $$
  \text{minimize } \sum_{e \in p} c_e \text{ over all paths } p \text{ from } s \text{ to } t.
  $$

- **Example:**

  $s$ $\xrightarrow{c_a = 4}$ $i$ $\xrightarrow{c_d = 12}$ $t$ $\xrightarrow{c_e = 5}$ $i$ $\xrightarrow{c_b = 3}$ $s$
Example: Buying a Path in a Network (ctd.)

- For $G = (V, E)$ and $e \in E$ let $G \setminus e = (V, E \setminus \{e\})$.
- VGC mechanism with Clarke pivot function:

  $$h_e(v_{-e}) = \max_{p' \in G \setminus e} \sum_{e' \in p'} -c_{e'}$$

  i.e., the cost of the cheapest path from $s$ to $t$ in $G \setminus e$.
  (Assume that $G$ is 2-connected, s.t. such $p'$ exists.)
- Payment functions: for chosen path $p = f(v_1, \ldots, v_n)$,

  $$p_e(v_1, \ldots, v_n) = h_e(v_{-e}) - \sum_{e \notin e' \in p} -c_{e'}.$$

- Case 1: $e \notin p$. Then $p_e(v_1, \ldots, v_n) = 0$.
- Case 2: $e \in p$. Then

  $$p_e(v_1, \ldots, v_n) = \max_{p' \in G \setminus e} \sum_{e' \in p'} -c_{e'} - \sum_{e \notin e' \in p} -c_{e'}.$$
Example: Buying a Path in a Network (ctd.)

Example:

Cost along $b$ and $e$: 8
Cost without $e$: 3
Cost of cheapest path without $e$: 15 (along $b$ and $d$)
Difference is payment: $-15 - (-3) = -12$

I.e., owner of arc $e$ gets payed 12 for using his arc.

Note: Alternative path after deletion of $e$ does not necessarily differ from original path at only one position. Could be totally different.
Summary

- New preference model: with money.
- VCG mechanisms generalize Vickrey auctions.
- VCG mechanisms are incentive compatible mechanisms maximizing social welfare.
- With Clarke pivot rule: even no positive transfers and individually rational (if nonnegative valuations).
- Various application areas.