Motivation

- Preference relations $\prec$ contain no information about “by how much” one candidate is preferred.
- Idea: Use money to measure this.
- Use money also for transfers between players “for compensation”.

June 23rd, 2016
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Setting

Formalization:

- Set of alternatives $A$.
- Set of $n$ players $I$.
- Valuation functions $v_i : A \to \mathbb{R}$ such that $v_i(a)$ denotes the value player $i$ assigns to alternative $a$.
- Payment functions specifying amount $p_i \in \mathbb{R}$ that player $i$ pays.
- Utility of player $i$: $u_i(a) = v_i(a) - p_i$. 
1 Second Price Auctions
Second Price Auctions

Second price auctions:

- There are $n$ players bidding for a single item.
- Player $i$’s private valuations of item: $w_i$.
- Desired outcome: Player with highest private valuation wins bid.
- Players should reveal their valuations truthfully.
- Winner $i$ pays price $p^*$ and has utility $w_i - p^*$.
- Non-winners pay nothing and have utility 0.
Second Price Auctions

Formally:

- \( A = N \)

- \( v_i(a) = \begin{cases} w_i & \text{if } a = i \\ 0 & \text{else} \end{cases} \)

What about payments? Say player \( i \) wins:

- \( p^* = 0 \) (winner pays nothing): bad idea, players would manipulate and publicly declare values \( w'_i \gg w_i \).
- \( p^* = w_i \) (winner pays his valuation): bad idea, players would manipulate and publicly declare values \( w'_i = w_i - \varepsilon \).
- better: \( p^* = \max_{j \neq i} w_j \) (winner pays second highest bid).
Vickrey Auction

Definition (Vickrey Auction)
The winner of the Vickrey Auction (aka second price auction) is the player $i$ with the highest declared value $w_i$. He has to pay the second highest declared bid $p^* = \max_{j \neq i} w_j$.

Proposition (Vickrey)
Let $i$ be one of the players and $w_i$ his valuation for the item, $u_i$ his utility if he truthfully declares $w_i$ as his valuation of the item, and $u'_i$ his utility if he falsely declares $w'_i$ as his valuation of the item. Then $u_i \geq u'_i$.

Proof
See
2 Incentive Compatible Mechanisms
Incentive Compatible Mechanisms

- **Idea**: Generalization of Vickrey auctions.
- Preferences modeled as functions \( v_i : A \rightarrow \mathbb{R} \).
- Let \( V_i \) be the space of all such functions for player \( i \).
- Unlike for social choice functions: Not only decide about chosen alternative, but also about payments.
Mechanisms

Definition (Mechanism)

A mechanism $\langle f, p_1, \ldots, p_n \rangle$ consists of

- a social choice function $f : V_1 \times \cdots \times V_n \to A$ and
- for each player $i$, a payment function $p_i : V_1 \times \cdots \times V_n \to \mathbb{R}$.

Definition (Incentive Compatibility)

A mechanism $\langle f, p_1, \ldots, p_n \rangle$ is called incentive compatible if for each player $i = 1, \ldots, n$, for all preferences $v_1 \in V_1, \ldots, v_n \in V_n$ and for each preference $v'_i \in V_i$,

$$v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}).$$
3 VCG Mechanisms

- Clarke Pivot Rule
- Examples
VCG Mechanisms

- If \( \langle f, p_1, \ldots, p_n \rangle \) is incentive compatible, truthfully declaring one's preference is dominant strategy.

- The Vickrey-Clarke-Groves mechanism is an incentive compatible mechanism that maximizes "social welfare", i.e., the sum over all individual utilities \( \sum_{i=1}^{n} v_i(a) \).

- Idea: Reflect other players' utilities in payment functions, align all players' incentives with goal of maximizing social welfare.
VCG Mechanisms

Definition (Vickrey-Clarke-Groves mechanism)

A mechanism $\langle f, p_1, \ldots, p_n \rangle$ is called a Vickrey-Clarke-Groves mechanism (VCG mechanism) if

1. $f(v_1, \ldots, v_n) \in \arg\max_{a \in A} \sum_{i=1}^{n} v_i(a)$ for all $v_1, \ldots, v_n$ and
2. there are functions $h_1, \ldots, h_n$ with $h_i : V_{-i} \rightarrow \mathbb{R}$ such that $p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \ldots, v_n))$ for all $i = 1, \ldots, n$ and $v_1, \ldots, v_n$.

Note: $h_i(v_{-i})$ independent of player $i$’s declared preference $\Rightarrow h_i(v_{-i}) = c$ constant from player $i$’s perspective.

Utility of player $i$ $\quad = v_i(f(v_1, \ldots, v_n)) + \sum_{j \neq i} v_j(f(v_1, \ldots, v_n)) - c = \sum_{j=1}^{n} v_j(f(v_1, \ldots, v_n)) - c = \text{social welfare} - c$. 
Theorem (Vickrey-Clarke-Groves)

Every VCG mechanism is incentive compatible.

Proof

Let $i, v_{-i}, v_i$ and $v'_i$ be given. Show: Declaring true preference $v_i$ dominates declaring false preference $v'_i$.

Let $a = f(v_i, v_{-i})$ and $a' = f(v'_i, v_{-i})$.

Utility player $i = \begin{cases} v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i}) & \text{if declaring } v_i \\ v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) & \text{if declaring } v'_i \end{cases}$

Alternative $a = f(v_i, v_{-i})$ maximizes social welfare

$\Rightarrow v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a').$

$\Rightarrow v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}). \quad \square$
Clarke Pivot Rule

- So far: payment functions $p_i$ and functions $h_i$ unspecified.
- One possibility: $h_i(v_{-i}) = 0$ for all $h_i$ and $v_{-i}$.
  Drawback: Too much money distributed among players (more that necessary).
- Further requirements:
  - Players should pay at most as much as they value the outcome.
  - Players should only pay, never receive money.
Definition (individual rationality)
A mechanism is individually rational if all players always get a nonnegative utility, i.e., if for all $i = 1, \ldots, n$ and all $v_1, \ldots, v_n$,

$$v_i(f(v_1, \ldots, v_n)) - p_i(v_1, \ldots, v_n) \geq 0.$$ 

Definition (positive transfers)
A mechanism has no positive transfers if no player is ever paid money, i.e., for all preferences $v_1, \ldots, v_n$,

$$p_i(v_1, \ldots, v_n) \geq 0.$$
Clarke Pivot Function

Definition (Clarke pivot function)

The Clarke pivot function is the function

\[ h_i(v_{-i}) = \max_{b \in A} \sum_{j \neq i} v_j(b). \]

This leads to payment functions

\[ p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \]

for \( a = f(v_1, \ldots, v_n) \).

Player \( i \) pays the difference between what the other players could achieve without him and what they achieve with him.

Each player internalizes the externalities he causes.
Clarke Pivot Function

Example

- Players \( I = \{1, 2\} \), alternatives \( A = \{a, b\} \).
- Values: \( v_1(a) = 10, v_1(b) = 2, v_2(a) = 9 \) and \( v_2(b) = 15 \).
- Without player 1: \( b \) best, since \( v_2(b) = 15 > 9 = v_2(a) \).
- With player 1: \( a \) best, since
  \[
  v_1(a) + v_2(a) = 10 + 9 = 19 > 17 = 2 + 15 = v_1(b) + v_2(b). 
  \]
- With player 1, other players (i.e., player 2) lose
  \( v_2(b) - v_2(a) = 6 \) units of utility.

\[\Rightarrow\text{ Clarke pivot function } h_1(v_2) = 15\]

\[\Rightarrow\text{ payment function }\]

\[
p_1(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq 1} v_j(b) - \sum_{j \neq 1} v_j(a) = 15 - 9 = 6.
\]
Lemma (Clarke pivot rule)

A VCG mechanism with Clarke pivot functions has no positive transfers. If \( v_i(a) \geq 0 \) for all \( i = 1, \ldots, n \), \( v_i \in V_i \) and \( a \in A \), then the mechanism is also individually rational.

Proof

Let \( a = f(v_1, \ldots, v_n) \) be the alternative maximizing \( \sum_{j=1}^{n} v_j(a) \), and \( b \) the alternative maximizing \( \sum_{j \neq i} v_j(b) \).

Utility of player \( i \): \( u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \).

Payment function for \( i \): \( p_i(v_1, \ldots, v_n) = \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \).

Since \( b \) maximizes \( \sum_{j \neq i} v_j(b) \): \( p_i(v_1, \ldots, v_n) \geq 0 \) (no positive transfers).
Clarke Pivot Rule

Proof (ctd.)

Individual rationality: Since $v_i(b) \geq 0$,

$$u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_{j=1}^{n} v_j(a) - \sum_{j=1}^{n} v_j(b).$$

Since $a$ maximizes $\sum_{j=1}^{n} v_j(a)$,

$$\sum_{j=1}^{n} v_j(a) \geq \sum_{j=1}^{n} v_j(b)$$

and hence $u_i \geq 0$.

Therefore, the mechanism is also individually rational. \qed
Vickrey Auction as a VCG Mechanism

- $A = N$. Valuations: $w_i$, $v_a(a) = w_a$, $v_i(a) = 0$ ($i \neq a$).
- $a$ maximizes social welfare $\sum_{i=1}^{n} v_i(a)$ iff $a$ maximizes $w_a$.
- Let $a = f(v_1, \ldots, v_n) = \arg\max_{j \in A} w_j$ be the highest bidder.
- Payments: $p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$.
- But $\max_{b \in A} \sum_{j \neq i} v_j(b) = \max_{b \in A \setminus \{i\}} w_b$.
- Winner pays value of second highest bid:
  \[
  p_a(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq a} v_j(b) - \sum_{j \neq a} v_j(a)
  = \max_{b \in A \setminus \{a\}} w_b - 0 = \max_{b \in A \setminus \{a\}} w_b.
  \]
- Non-winners pay nothing: For $i \neq a$,
  \[
  p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)
  = \max_{b \in A \setminus \{i\}} w_b - w_a = w_a - w_a = 0.
  \]
Example: Bilateral Trade

- **Seller** offers item he values with \(0 \leq w_s \leq 1\).
- **Potential buyer** values item with \(0 \leq w_b \leq 1\).
- **Alternatives** \(A = \{\text{trade, no-trade}\}\).
- **Valuations:**
  
  \[
  v_s(\text{no-trade}) = 0, \quad v_s(\text{trade}) = -w_s, \\
  v_b(\text{no-trade}) = 0, \quad v_b(\text{trade}) = w_b.
  \]

- **VCG mechanism** maximizes \(v_s(a) + v_b(a)\).
- We have
  
  \[
  v_s(\text{trade}) + v_b(\text{trade}) = w_b - w_s, \\
  v_s(\text{no-trade}) + v_b(\text{no-trade}) = 0
  \]

  i.e., **trade** maximizes social welfare iff \(w_b \geq w_s\).
Example: Bilateral Trade (ctd.)

- **Requirement:** if *no-trade* is chosen, neither player pays anything:

  \[ p_s(v_s, v_b) = p_b(v_s, v_b) = 0. \]

- To that end, choose Clarke pivot function for **buyer**:

  \[ h_b(v_s) = \max_{a \in A} v_s(a). \]

- For **seller**: Modify Clarke pivot function by an additive constant and set

  \[ h_s(v_b) = \max_{a \in A} v_b(a) - w_b. \]
Example: Bilateral Trade (ctd.)

- For alternative \textit{no-trade},

\[
p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\textit{no-trade})
\]

\[
= w_b - w_b - 0 = 0 \quad \text{and}
\]

\[
p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\textit{no-trade})
\]

\[
= 0 - 0 = 0.
\]

- For alternative \textit{trade},

\[
p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\textit{trade})
\]

\[
= w_b - w_b - w_b = -w_b \quad \text{and}
\]

\[
p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\textit{trade})
\]

\[
= 0 + w_s = w_s.
\]
Example: Bilateral Trade (ctd.)

- Because \( w_b \geq w_s \), the seller gets at least as much as the buyer pays, i.e., the mechanism subsidizes the trade.
- Without subsidies, no incentive compatible bilateral trade possible.

**Note:** Buyer and seller can exploit the system by colluding.
Example: Public Project

- Project costs $C$ units.
- Each citizen $i$ privately values the project at $w_i$ units.
- Government will undertake project if $\sum_i w_i > C$.
- Alternatives: $A = \{\text{project, no-project}\}$.
- Valuations:
  
  $$v_G(\text{project}) = -C, \quad v_G(\text{no-project}) = 0,$$
  $$v_i(\text{project}) = w_i, \quad v_i(\text{no-project}) = 0.$$

- VCG mechanism with Clarke pivot rule: for each citizen $i$,

  $$h_i(v_{-i}) = \max_{a \in A} \left( \sum_{j \neq i} v_j(a) + v_G(a) \right)$$

  $$= \begin{cases} 
  \sum_{j \neq i} w_j - C, & \text{if } \sum_{j \neq i} w_j > C \\
  0, & \text{otherwise.}
  \end{cases}$$
Example: Public Project (ctd.)

Citizen $i$ pivotal if $\sum_j w_j > C$ and $\sum_{j \neq i} w_j \leq C$.

Payment function for citizen $i$:

$$p_i(v_{1..n}, v_G) = h_i(v_{-i}) - \left( \sum_{j \neq i} v_j(f(v_{1..n}, v_G)) + v_G(f(v_{1..n}, v_G)) \right)$$

Case 1: Project undertaken, $i$ pivotal:

$$p_i(v_{1..n}, v_G) = 0 - \left( \sum_{j \neq i} w_j - C \right) = C - \sum_{j \neq i} w_j$$

Case 2: Project undertaken, $i$ not pivotal:

$$p_i(v_{1..n}, v_G) = \left( \sum_{j \neq i} w_j - C \right) - \left( \sum_{j \neq i} w_j - C \right) = 0$$

Case 3: Project not undertaken:

$$p_i(v_{1..n}, v_G) = 0$$
Example: Public Project (ctd.)

- I.e., citizen \( i \) pays nonzero amount
  
  \[ C - \sum_{j \neq i} w_j \]

  only if he is pivotal.

- He pays difference between value of project to fellow citizens and cost \( C \), in general less than \( w_i \).

- Generally,
  
  \[ \sum_{i} p_i(\text{project}) \leq C \]

  i.e., project has to be subsidized.
Example: Buying a Path in a Network

- Communication network modeled as $G = (V, E)$.
- Each link $e \in E$ owned by different player $e$.
- Each link $e \in E$ has cost $c_e$ if used.
- **Objective:** procure communication path from $s$ to $t$.
- **Alternatives:** $A = \{p \mid p$ path from $s$ to $t\}$.
- **Valuations:** $v_e(p) = -c_e$, if $e \in p$, and $v_e(p) = 0$, if $e \notin p$.
- **Maximizing social welfare:**
  
  minimize $\sum_{e \in p} c_e$ over all paths $p$ from $s$ to $t$.

- **Example:**

  $c_a = 4$
  $c_b = 3$
  $c_d = 12$
  $c_e = 5$
Example: Buying a Path in a Network (ctd.)

For $G = (V, E)$ and $e \in E$ let $G \setminus e = (V, E \setminus \{e\})$.

VGC mechanism with Clarke pivot function:

$$h_e(v_e) = \max_{p' \in G \setminus e} \sum e' \in p' - c_{e'}$$

i.e., the cost of the cheapest path from $s$ to $t$ in $G \setminus e$.

(Assume that $G$ is 2-connected, s.t. such $p'$ exists.)

Payment functions: for chosen path $p = f(v_1, \ldots, v_n)$,

$$p_e(v_1, \ldots, v_n) = h_e(v_e) - \sum_{e \neq e' \in p} -c_{e'}.$$  

- Case 1: $e \notin p$. Then $p_e(v_1, \ldots, v_n) = 0$.
- Case 2: $e \in p$. Then

$$p_e(v_1, \ldots, v_n) = \max_{p' \in G \setminus e} \sum e' \in p' - c_{e'} - \sum_{e \neq e' \in p} -c_{e'}.$$
Example: Buying a Path in a Network (ctd.)

Example:

\[
\begin{align*}
  c_a &= 4 \\
  c_b &= 3 \\
  c_d &= 12 \\
  c_e &= 5
\end{align*}
\]

Cost along \( b \) and \( e \): 8
Cost without \( e \): 3
Cost of cheapest path without \( e \): 15 (along \( b \) and \( d \))
Difference is payment: \(-15 - (-3) = -12\)
I.e., owner of arc \( e \) gets payed 12 for using his arc.

Note: Alternative path after deletion of \( e \) does not necessarily differ from original path at only one position. Could be totally different.
Summary

- New preference model: with **money**.
- VCG mechanisms generalize **Vickrey auctions**.
- VCG mechanisms are incentive compatible mechanisms maximizing social welfare.
- With Clarke pivot rule: even no positive transfers and individually rational (if nonnegative valuations).
- Various application areas.