Motivation
Motivation

- **So far:** All players move *simultaneously*, and then the outcome is determined.
- **Often in practice:** Several moves in *sequence* (e.g. in chess).
  ~~~ cannot be directly reflected by strategic games.
- **Extensive games** (with perfect information) reflect such situations by modeling games as *game trees*.
- **Idea:** Players have several decision points where they can decide how to play.
- **Strategies:** Mappings from decision points in the game tree to actions to be played.
Definitions
## Definition (Extensive game with perfect information)

An **extensive game with perfect information** is a tuple

\[ \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \]

that consists of:

- A finite non-empty set \( N \) of players.
- A set \( H \) of (finite or infinite) sequences, called **histories**, such that
  - the empty sequence \( \langle \rangle \in H \),
  - \( H \) is closed under prefixes: if \( \langle a^1, \ldots, a^k \rangle \in H \) for some \( k \in \mathbb{N} \cup \{\infty\} \), and \( l < k \), then also \( \langle a^1, \ldots, a^l \rangle \in H \), and
  - \( H \) is closed under limits: if for some infinite sequence \( \langle a^i \rangle_{i=1}^{\infty} \), we have \( \langle a^i \rangle_{i=1}^{k} \in H \) for all \( k \in \mathbb{N} \), then \( \langle a^i \rangle_{i=1}^{\infty} \in H \).

All infinite histories and all histories \( \langle a^i \rangle_{i=1}^{k} \in H \), for which there is no \( a^{k+1} \) such that \( \langle a^i \rangle_{i=1}^{k+1} \in H \) are called **terminal histories** \( Z \). Components of a history are called **actions**.
A player function $P : H \setminus Z \to N$ that determines which player’s turn it is to move after a given nonterminal history.

For each player $i \in N$, a utility function (or payoff function) $u_i : Z \to \mathbb{R}$ defined on the set of terminal histories.

The game is called finite, if $H$ is finite. It has a finite horizon, if the length of histories is bounded from above.

Assumption: All ingredients of $\Gamma$ are common knowledge amongst the players of the game.

Terminology: In the following, we will simply write extensive games instead of extensive games with perfect information.
Example (Division game)

- Two identical objects should be divided among two players.
- Player 1 proposes an allocation.
- Player 2 agrees or rejects.
  - On agreement: Allocation as proposed.
  - On rejection: Nobody gets anything.
Example (Division game, formally)

\[ \begin{align*} 
N &= \{1, 2\} \\
H &= \{\langle\rangle, \langle(2, 0)\rangle, \langle(1, 1)\rangle, \langle(0, 2)\rangle, \langle(2, 0), y\rangle, \langle(2, 0), n\rangle, \ldots\} \\
P(\langle\rangle) &= 1, \ P(h) = 2 \text{ for all } h \in H \setminus Z \text{ with } h \neq \langle\rangle \\
u_1(\langle(2, 0), y\rangle) &= 2, \ u_2(\langle(2, 0), y\rangle) = 0, \text{ etc.} 
\end{align*} \]
Notation:

Let $h = \langle a^1, \ldots, a^k \rangle$ be a history, and $a$ an action.

- Then $(h, a)$ is the history $\langle a^1, \ldots, a^k, a \rangle$.
- If $h' = \langle b^1, \ldots, b^\ell \rangle$, then $(h, h')$ is the history $\langle a^1, \ldots, a^k, b^1, \ldots, b^\ell \rangle$.
- The set of actions from which player $P(h)$ can choose after a history $h \in H \setminus Z$ is written as

$$A(h) = \{ a \mid (h, a) \in H \}.$$
Definition (Strategy in an extensive game)

A strategy of a player $i$ in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is a function $s_i$ that assigns to each nonterminal history $h \in H \setminus Z$ with $P(h) = i$ an action $a \in A(h)$. The set of strategies of player $i$ is denoted as $S_i$.

**Remark:** Strategies require us to assign actions to histories $h$, even if it is clear that they will never be played (e.g., because $h$ will never be reached because of some earlier action).

**Notation (for finite games):** A strategy for a player is written as a string of actions at decision nodes as visited in a breadth-first order.
Example (Strategies in an extensive game)

- Strategies for player 1: $AE$, $AF$, $BE$ and $BF$
- Strategies for player 2: $C$ and $D$. 
**Definition (Outcome)**

The **outcome** \( O(s) \) of a strategy profile \( s = (s_i)_{i \in N} \) is the (possibly infinite) terminal history \( h = \langle a^i \rangle_{i=1}^k \), with \( k \in \mathbb{N} \cup \{\infty\} \), such that for all \( \ell \in \mathbb{N} \) with \( 0 \leq \ell < k \),

\[
S_P(\langle a^1, \dots, a^\ell \rangle)(\langle a^1, \ldots, a^\ell \rangle) = a^{\ell+1}.
\]

**Example (Outcome)**

- \( P(\langle \rangle) = 1 \)
- \( P(\langle A \rangle) = 2 \)
- \( P(\langle A, C \rangle) = 1 \)
- \( P(\langle A, F \rangle) = 1 \)
- \( P(\langle A, E \rangle) = 1 \)
- \( O(AF, C) = \langle A, C, F \rangle \)
- \( O(AE, D) = \langle A, D \rangle \).
Solution Concepts
Definition (Nash equilibrium in an extensive game)

A Nash equilibrium in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is a strategy profile $s^*$ such that for every player $i \in N$ and for all strategies $s_i \in S_i$,

$$u_i(O(s^*_i, s_i^*)) \geq u_i(O(s^*_i, s_i)).$$
**Definition (Induced strategic game)**

The strategic game $G$ induced by an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is defined by $G = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$, where

- $A'_i = S_i$ for all $i \in N$, and
- $u'_i(a) = u_i(O(a))$ for all $i \in N$.

**Proposition**

The Nash equilibria of an extensive game $\Gamma$ are exactly the Nash equilibria of the induced strategic game $G$ of $\Gamma$. 
Remarks:

- Each extensive game can be transformed into a strategic game, but the resulting game can be exponentially larger.
- The other direction does not work, because in extensive games, we do not have simultaneous actions.
Example (Empty threat)

Extensive game:

Strategic form:

Nash equilibria: \((B, L)\) and \((T, R)\).

However, \((B, L)\) is not realistic:

- Player 1 plays \(B\), “fearing” response \(L\) to \(T\).
- But player 2 would never play \(L\) in the extensive game.

\(\rightarrow \) \((B, L)\) involves “empty threat”.
Subgames

Idea: Exclude empty threats.

How? Demand that a strategy profile is not only a Nash equilibrium in the strategic form, but also in every subgame.

**Definition (Subgame)**

A subgame of an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \), starting after history \( h \), is the game \( \Gamma(h) = \langle N, H|_h, P|_h, (u_i|_h)_{i \in N} \rangle \), where

- \( H|_h = \{ h' \mid (h, h') \in H \} \),
- \( P|_h(h') = P(h, h') \) for all \( h' \in H|_h \), and
- \( u_i|_h(h') = u_i(h, h') \) for all \( h' \in H|_h \).
Definition (Strategy in a subgame)

Let $\Gamma$ be an extensive game and $\Gamma(h)$ a subgame of $\Gamma$ starting after some history $h$.

For each strategy $s_i$ of $\Gamma$, let $s_i|_h$ be the strategy induced by $s_i$ for $\Gamma(h)$. Formally, for all $h' \in H|_h$,

$$s_i|_h(h') = s_i(h, h').$$

The outcome function of $\Gamma(h)$ is denoted by $O_h$. 
Definition (Subgame-perfect equilibrium)

A strategy profile \( s^* \) in an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \) is a subgame-perfect equilibrium if and only if for every player \( i \in N \) and every nonterminal history \( h \in H \setminus Z \) with \( P(h) = i \),

\[
 u_i |_h (O_h(s^*_{-i} | h, s^*_i | h)) \geq u_i |_h (O_h(s^*_{-i} | h, s_i))
\]

for every strategy \( s_i \in S_i \) in subgame \( \Gamma(h) \).
Subgame-Perfect Equilibria

Two Nash equilibria:

- \((T, R)\): subgame-perfect, because:
  - In history \(h = \langle T \rangle\): subgame-perfect.
  - In history \(h = \langle \rangle\): player 1 obtains utility 1 when choosing \(B\) and utility of 2 when choosing \(T\).

- \((B, L)\): not subgame-perfect, since \(L\) does not maximize the utility of player 2 in history \(h = \langle T \rangle\).
Example (Subgame-perfect equilibria in division game)

Equilibria in subgames:
- in $\Gamma(\langle 2, 0 \rangle)$: $y$ and $n$
- in $\Gamma(\langle 1, 1 \rangle)$: only $y$
- in $\Gamma(\langle 0, 2 \rangle)$: only $y$
- in $\Gamma(\langle \rangle)$: $\langle (2, 0), yyy \rangle$
  and $\langle (1, 1), nyy \rangle$

Nash equilibria (red: empty threat):
- $\langle (2, 0), yyy \rangle$, $\langle (2, 0), yyn \rangle$, $\langle (2, 0), yny \rangle$, $\langle (2, 0), ynn \rangle$,
  $\langle (2, 0), nny \rangle$, $\langle (2, 0), nnn \rangle$,
- $\langle (1, 1), nyy \rangle$, $\langle (1, 1), nyn \rangle$,
- $\langle (0, 2), nny \rangle$, $\langle (0, 2), nnn \rangle$. 
One-Deviation Property
Motivation

- **Existence:**
  - Does every extensive game have a subgame-perfect equilibrium?
  - If not, which extensive games do have a subgame-perfect equilibrium?

- **Computation:**
  - If a subgame-perfect equilibrium exists, how to compute it?
  - How complex is that computation?
Positive case (a subgame-perfect equilibrium exists):

- **Step 1**: Show that it suffices to consider local deviations from strategies (for finite-horizon games).
- **Step 2**: Show how to systematically explore such local deviations to find a subgame-perfect equilibrium (for finite games).
Step 1: One-Deviation Property

**Definition**
Let $\Gamma$ be a finite-horizon extensive game. Then $\ell(\Gamma)$ denotes the length of the longest history of $\Gamma$. 
Step 1: One-Deviation Property

Definition (One-deviation property)

A strategy profile $s^*$ in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ satisfies the one-deviation property if and only if for every player $i \in N$ and every nonterminal history $h \in H \setminus Z$ with $P(h) = i$,

$$u_i|_h(O_h(s_{-i}^*, s_i^*|h)) \geq u_i|_h(O_h(s_{-i}^*, s_i|h))$$

for every strategy $s_i \in S_i$ in subgame $\Gamma(h)$ that differs from $s_i^*|_h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

Note: Without the highlighted parts, this is just the definition of subgame-perfect equilibria!
Step 1: One-Deviation Property

Lemma
Let $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ be a finite-horizon extensive game. Then a strategy profile $s^*$ is a subgame-perfect equilibrium of $\Gamma$ if and only if it satisfies the one-deviation property.

Proof
- $(\Rightarrow)$ Clear.
- $(\Leftarrow)$ By contradiction:
  
  Suppose that $s^*$ is not a subgame-perfect equilibrium. Then there is a history $h$ and a player $i$ such that $s_i$ is a profitable deviation for player $i$ in subgame $\Gamma(h)$.

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Step 1: One-Deviation Property

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Step 1: One-Deviation Property

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Step 1: One-Deviation Property

Proof (ctd.)

\(\Longleftrightarrow \) ... WLOG, the number of histories \(h'\) with \(s_i(h') \neq s_i^*|_h(h')\) is at most \(\ell(\Gamma(h))\) and hence finite (finite horizon assumption!), since deviations not on resulting outcome path are irrelevant.

Illustration: strategies \(s_1^*|_h = AGILN\) and \(s_2^*|_h = CF\) red:

\[
P(h) = 1
\]
Proof (ctd.)

\[ \iff \ldots \text{WLOG, the number of histories } h' \text{ with } s_i(h') \neq s_i^*|_h(h') \text{ is at most } \ell(\Gamma(h)) \text{ and hence finite (finite horizon assumption!), since deviations not on resulting outcome path are irrelevant.} \]

Illustration: strategies \( s_1^*|_h = AGILN \) and \( s_2^*|_h = CF \) red:
Step 1: One-Deviation Property

Proof (ctd.)

(⇐) ... Illustration for WLOG assumption: Assume $s_1 = BHKMO$ (blue) profitable deviation:

$$P(h) = 1$$

Then only $B$ and $O$ really matter.
Proof (ctd.)

\[ \left( \leftarrow \right) \ldots \text{Illustration for WLOG assumption: And hence } \hat{s}_1 = \text{BGIL0 (blue)} \text{ also profitable deviation:} \]

\[ P(h) = 1 \]
Choose profitable deviation $s_i$ in $\Gamma(h)$ with minimal number of deviation points (such $s_i$ must exist).

Let $h^*$ be the longest history in $\Gamma(h)$ with $s_i(h^*) \neq s_i^*|_h(h^*)$, i.e., “deepest” deviation point for $s_i$.

Then in $\Gamma(h, h^*)$, $s_i|_{h^*}$ differs from $s_i^*|_{(h, h^*)}$ only in the initial history.

Moreover, $s_i|_{h^*}$ is a profitable deviation in $\Gamma(h, h^*)$, since $h^*$ is the longest history in $\Gamma(h)$ with $s_i(h^*) \neq s_i^*|_h(h^*)$.

So, $\Gamma(h, h^*)$ is the desired subgame where a one-step deviation is sufficient to improve utility.
Step 1: One-Deviation Property

Proof (ctd.)

\[ \text{Choose profitable deviation } s_i \text{ in } \Gamma(h) \text{ with minimal number of deviation points (such } s_i \text{ must exist).} \]

Let \( h^* \) be the longest history in \( \Gamma(h) \) with \( s_i(h^*) \neq s_i^*|_{h(h^*)} \), i.e., “deepest” deviation point for \( s_i \).

Then in \( \Gamma(h, h^*) \), \( s_i|_{h^*} \) differs from \( s_i^*|_{(h, h^*)} \) only in the initial history.

Moreover, \( s_i|_{h^*} \) is a profitable deviation in \( \Gamma(h, h^*) \), since \( h^* \) is the longest history in \( \Gamma(h) \) with \( s_i(h^*) \neq s_i^*|_{h(h^*)} \).

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\[ \leftarrow\rightarrow \ldots \]

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So, \( \Gamma(h, h^*) \) is the desired subgame where a one-step deviation is sufficient to improve utility.
To show that \((AHI, CE)\) is a subgame-perfect equilibrium, it suffices to check these deviating strategies:

**Player 1:**
- \(G\) in subgame \(\Gamma(\langle A, C \rangle)\)
- \(K\) in subgame \(\Gamma(\langle B, F \rangle)\)
- \(BHI\) in \(\Gamma\)

**Player 2:**
- \(D\) in subgame \(\Gamma(\langle A \rangle)\)
- \(F\) in subgame \(\Gamma(\langle B \rangle)\)

In particular, e.g., no need to check if strategy \(BGK\) of player 1 is profitable in \(\Gamma\).
The corresponding proposition for infinite-horizon games does not hold.

Counterexample (one-player case):

Strategy $s_i$ with $s_i(h) = S$ for all $h \in H \setminus Z$

- satisfies one deviation property, but
- is not a subgame-perfect equilibrium, since it is dominated by $s_i^*$ with $s_i^*(h) = C$ for all $h \in H \setminus Z$. 
Kuhn’s Theorem
Step 2: Kuhn’s Theorem

Theorem (Kuhn)
Every finite extensive game has a subgame-perfect equilibrium.

Proof idea:
- Proof is constructive and builds a subgame-perfect equilibrium bottom-up (aka backward induction).
- For those familiar with the Foundations of AI lecture: generalization of Minimax algorithm to general-sum games with possibly more than two players.
Step 2: Kuhn’s Theorem

Example

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</tbody>
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Motivation
Definitions
Solution Concepts
One-Deviation Property
Kuhn’s Theorem
Two Extensions
Summary
Step 2: Kuhn’s Theorem

Example

\[s_2(\langle A \rangle) = C, \quad t_1(\langle A \rangle) = 1, \quad t_2(\langle A \rangle) = 5\]
Step 2: Kuhn’s Theorem

Example

\[ s_2(\langle A \rangle) = C \]
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\[ t_2(\langle B \rangle) = 8 \]
Step 2: Kuhn’s Theorem

Example

\[ s_2(\langle A \rangle) = C \quad t_1(\langle A \rangle) = 1 \quad t_2(\langle A \rangle) = 5 \]
\[ s_2(\langle B \rangle) = F \quad t_1(\langle B \rangle) = 0 \quad t_2(\langle B \rangle) = 8 \]
\[ s_1(\langle \rangle) = A \quad t_1(\langle \rangle) = 1 \quad t_2(\langle \rangle) = 5 \]
A bit more formally:

Proof

Let $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ be a finite extensive game.

Construct a subgame-perfect equilibrium by induction on $\ell(\Gamma(h))$ for all subgames $\Gamma(h)$. In parallel, construct functions $t_i : H \rightarrow \mathbb{R}$ for all players $i \in N$ s.t. $t_i(h)$ is the payoff for player $i$ in a subgame-perfect equilibrium in subgame $\Gamma(h)$.

Base case: If $\ell(\Gamma(h)) = 0$, then $t_i(h) = u_i(h)$ for all $i \in N$.

...
Step 2: Kuhn’s Theorem

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Step 2: Kuhn’s Theorem

A bit more formally:

**Proof**

Let \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \) be a finite extensive game.

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...
Step 2: Kuhn’s Theorem

Proof (ctd.)

**Inductive case:** If $t_i(h)$ already defined for all $h \in H$ with $\ell(\Gamma(h)) \leq k$, consider $h^* \in H$ with $\ell(\Gamma(h^*)) = k + 1$ and $P(h^*) = i$.

For all $a \in A(h^*)$, $\ell(\Gamma(h^*,a)) \leq k$, let

$$s_i(h^*) := \arg\max_{a \in A(h^*)} t_i(h^*,a)$$

and

$$t_j(h^*) := t_j(h^*,s_i(h^*))$$

for all players $j \in N$.

Inductively, we obtain a strategy profile $s$ that satisfies the one-deviation property.

With the one-deviation property lemma it follows that $s$ is a subgame-perfect equilibrium.
Step 2: Kuhn’s Theorem

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\[
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Step 2: Kuhn’s Theorem

- **In principle:** sample subgame-perfect equilibrium effectively computable using the technique from the above proof.

- **In practice:** often game trees not enumerated in advance, hence unavailable for backward induction.

- E.g., for branching factor $b$ and depth $m$, procedure needs time $O(b^m)$. 
Corresponding proposition for infinite games does not hold.

Counterexamples (both for one-player case):

A) finite horizon, infinite branching factor:

Infinitely many actions \( a \in A = [0, 1) \) with payoffs \( u_1(\langle a \rangle) = a \) for all \( a \in A \).

There exists no subgame-perfect equilibrium in this game.
B) infinite horizon, finite branching factor:

\[ u_1(\text{CCC} \ldots) = 0 \text{ and } u_1(\text{CC} \ldots \text{CS}) = n + 1. \]

No subgame-perfect equilibrium.
Step 2: Kuhn’s Theorem

Uniqueness:

Kuhn’s theorem tells us nothing about uniqueness of subgame-perfect equilibria. However, if no two histories get the same evaluation by any player, then the subgame-perfect equilibrium is unique.
Two Extensions
An extensive game with chance moves is a tuple \( \Gamma = \langle N, H, P, f_c, (u_i)_{i \in N} \rangle \), where

- \( N, A, H \) and \( u_i \) are defined as before,
- the player function \( P : H \setminus Z \to N \cup \{c\} \) can also take the value \( c \) for a chance node, and
- for each \( h \in H \setminus Z \) with \( P(h) = c \), the function \( f_c(\cdot | h) \) is a probability distribution on \( A(h) \) such that the probability distributions for all \( h \in H \) are independent of each other.
Intended meaning of chance moves: In chance node, an applicable action is chosen randomly with probability according to $f_c$.

Strategies: Defined as before.

Outcome: For a given strategy profile, the outcome is a probability distribution on the set of terminal histories.

Payoffs: For player $i$, $U_i$ is the expected payoff (with weights according to outcome probabilities).
Chance Moves

Example

\[ P(\langle A \rangle) = c \]
\[ P(\langle B \rangle) = c \]
\[ f_c(\langle A \rangle) = \frac{1}{2} \]
\[ f_c(\langle B \rangle) = \frac{1}{2} \]
\[ f_c(\langle C \rangle) = \frac{1}{3} \]
\[ f_c(\langle D \rangle) = \frac{2}{3} \]

\[ P(\langle B, F \rangle) = 2 \]
\[ P(\langle B, G \rangle) = 2 \]
\[ P(\langle B, C \rangle) = 2 \]

\[ f_c(\langle D \mid \langle A \rangle \rangle) = \frac{1}{2} \]
\[ f_c(\langle E \mid \langle A \rangle \rangle) = \frac{1}{2} \]
\[ f_c(\langle F \mid \langle B \rangle \rangle) = \frac{1}{3} \]
\[ f_c(\langle G \mid \langle B \rangle \rangle) = \frac{2}{3} \]

\[ (0, 6) \]
\[ (2, 2) \]
\[ (0, 3) \]
\[ (2, 2) \]
\[ (3, 3) \]
\[ (3, 3) \]
Remark:
The one-deviation property and Kuhn’s theorem still hold in the presence of chance moves. When proving Kuhn’s theorem, expected utilities have to be used.
An extensive game with simultaneous moves is a tuple \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \), where

- \( N, A, H \) and \( (u_i) \) are defined as before, and
- \( P : H \rightarrow 2^N \) assigns to each nonterminal history a set of players to move; for all \( h \in H \setminus Z \), there exists a family \( (A_i(h))_{i \in P(h)} \) such that

\[
A(h) = \{ a \mid (h,a) \in H \} = \prod_{i \in P(h)} A_i(h).
\]
Simultaneous Moves

- **Intended meaning of simultaneous moves:** All players from \( P(h) \) move simultaneously.
- **Strategies:** Functions \( s_i : h \mapsto a_i \) with \( a_i \in A_i(h) \).
- **Histories:** Sequences of vectors of actions.
- **Outcome:** Terminal history reached when tracing strategy profile.
- **Payoffs:** Utilities at outcome history.
Remark:

- The one-deviation property still holds for extensive game with perfect information and simultaneous moves.
- Kuhn’s theorem does not hold for extensive game with simultaneous moves.

**Example:** Matching Pennies can be viewed as extensive game with simultaneous moves. No Nash equilibrium/subgame-perfect equilibrium.

\[
\begin{array}{c|cc}
 & H & T \\
\hline
H & 1, -1 & -1, 1 \\
T & -1, 1 & 1, -1 \\
\end{array}
\]

\[\rightsquigarrow\text{ Need more sophisticated solution concepts (cf. mixed strategies). Not covered in this lecture.}\]
Simultaneous Moves
Example: Three-Person Cake Splitting Game

Setting:

- Three players have to split a cake fairly.
- Player 1 suggests a split: shares $x_1, x_2, x_3 \in [0, 1]$ s.t. $x_1 + x_2 + x_3 = 1$.
- Then players 2 and 3 simultaneously and independently decide whether to accept (“y”) or reject (“n”) the suggested splitting.
- If both accept, each player $i$ gets his allotted share (utility $x_i$). Otherwise, no player gets anything (utility 0).
Simultaneous Moves
Example: Three-Person Cake Splitting Game

Formally:

\[ N = \{1, 2, 3\} \]
\[ X = \{(x_1, x_2, x_3) \in [0, 1]^3 \mid x_1 + x_2 + x_3 = 1\} \]
\[ H = \{\{\}\} \cup \{\langle x \rangle \mid x \in X\} \cup \{\langle x, z \rangle \mid x \in X, z \in \{y, n\} \times \{y, n\}\} \]
\[ P(\{\}\) = \{1\} \]
\[ P(\langle x \rangle) = \{2, 3\} \text{ for all } x \in X \]
\[ u_i(\langle x, z \rangle) = \begin{cases} 0 & \text{if } z \in \{(y, n), (n, y), (n, n)\} \\ x_i & \text{if } z = (y, y). \end{cases} \text{ for all } i \in N \]
Simultaneous Moves
Example: Three-Person Cake Splitting Game

Subgame-perfect equilibria:

- **Subgames after legal split** \((x_1, x_2, x_3)\) by player 1:
  - NE \((y, y)\) (both accept)
  - NE \((n, n)\) (neither accepts)
  - If \(x_2 = 0\), NE \((n, y)\) (only player 3 accepts)
  - If \(x_3 = 0\), NE \((y, n)\) (only player 2 accepts)
Subgame-perfect equilibria (ctd.):

**Entire game:**

Let $s_2$ and $s_3$ be any two strategies of players 2 and 3 such that for all splits $x \in X$ the profile $(s_2(\langle x \rangle), s_3(\langle x \rangle))$ is one of the NEs from above.

Let $X_y = \{x \in X \mid s_2(\langle x \rangle) = s_3(\langle x \rangle) = y\}$ be the set of splits accepted under $s_2$ and $s_3$. Distinguish three cases:

- $X_y = \emptyset$ or $x_1 = 0$ for all $x \in X_y$. Then $(s_1, s_2, s_3)$ is a subgame-perfect equilibrium for any possible $s_1$.
- $X_y \neq \emptyset$ and there are splits $x_{\text{max}} = (x_1, x_2, x_3) \in X_y$ that maximize $x_1 > 0$. Then $(s_1, s_2, s_3)$ is a subgame-perfect equilibrium if and only if $s_1(\langle \rangle)$ is such a split $x_{\text{max}}$.
- $X_y \neq \emptyset$ and there are no splits $(x_1, x_2, x_3) \in X_y$ that maximize $x_1$. Then there is no subgame-perfect equilibrium, in which player 2 follows strategy $s_2$ and player 3 follows strategy $s_3$. 
Summary
For **finite-horizon extensive games**, it suffices to consider **local deviations** when looking for better strategies.

For infinite-horizon games, this is not true in general.

Every **finite extensive game** has a **subgame-perfect equilibrium**.

This does not generally hold for infinite games, no matter is game is infinite due to infinite branching factor or infinitely long histories (or both).

- **With chance moves**, one deviation property and Kuhn’s theorem still hold.
- **With simultaneous moves**, Kuhn’s theorem no longer holds.