Motivation

- So far: All players move simultaneously, and then the outcome is determined.
- Often in practice: Several moves in sequence (e.g. in chess).
  \( \rightarrow \) cannot be directly reflected by strategic games.
- Extensive games (with perfect information) reflect such situations by modeling games as game trees.
- Idea: Players have several decision points where they can decide how to play.
- Strategies: Mappings from decision points in the game tree to actions to be played.
Definition (Extensive game with perfect information)

An extensive game with perfect information is a tuple \( \Gamma = (N, H, P, (u_i)_{i \in N}) \) that consists of:

- A finite non-empty set \( N \) of players.
- A set \( H \) of (finite or infinite) sequences, called histories, such that
  - the empty sequence \( \langle \rangle \in H \),
  - \( H \) is closed under prefixes: if \( \langle a^1, \ldots, a^k \rangle \in H \) for some \( k \in \mathbb{N} \cup \{\infty\} \), and \( i < k \), then also \( \langle a^1, \ldots, a^i \rangle \in H \), and
  - \( H \) is closed under limits: if for some infinite sequence \( \langle a^i \rangle_{i=1}^{\infty} \), we have \( \langle a^i \rangle_{i=k}^k \in H \) for all \( k \in \mathbb{N} \), then \( \langle a^i \rangle_{i=1}^{\infty} \in H \).

All infinite histories and all histories \( \langle a^i \rangle_{i=1}^k \in H \), for which there is no \( a^i+1 \) such that \( \langle a^i \rangle_{i=1}^k \in H \) are called terminal histories \( Z \). Components of a history are called actions.

Assumption: All ingredients of \( \Gamma \) are common knowledge amongst the players of the game.

Terminology: In the following, we will simply write extensive games instead of extensive games with perfect information.

Example (Division game)

Two identical objects should be divided among two players.

- **Player 1 proposes** an allocation.
- **Player 2 agrees** or rejects.
  - On agreement: Allocation as proposed.
  - On rejection: Nobody gets anything.
**Extensive Games**

**Notation:**
Let \( h = \langle a^1, \ldots, a^k \rangle \) be a history, and \( a \) an action.
- Then \( (h, a) \) is the history \( \langle a^1, \ldots, a^k, a \rangle \).
- If \( h' = \langle b^1, \ldots, b'^{\ell} \rangle \), then \( (h, h') \) is the history \( \langle a^1, \ldots, a^k, b^1, \ldots, b'^{\ell} \rangle \).
- The set of actions from which player \( P(h) \) can choose after a history \( h \in H \setminus Z \) is written as \( A(h) = \{ a \mid (h, a) \in H \} \).

---

**Strategies**

**Definition (Strategy in an extensive game)**
A strategy of a player \( i \) in an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \) is a function \( s_i \) that assigns to each nonterminal history \( h \in H \setminus Z \) with \( P(h) = i \) an action \( a \in A(h) \). The set of strategies of player \( i \) is denoted as \( S_i \).

**Remark:** Strategies require us to assign actions to histories \( h \), even if it is clear that they will never be played (e.g., because \( h \) will never be reached because of some earlier action).

**Notation (for finite games):** A strategy for a player is written as a string of actions at decision nodes as visited in a breadth-first order.

---

**Outcome**

**Definition (Outcome)**
The outcome \( O(s) \) of a strategy profile \( s = (s_i)_{i \in N} \) is the (possibly infinite) terminal history \( h = \langle a^i \rangle_{i=1}^k \), with \( k \in \mathbb{N} \cup \{ \infty \} \), such that for all \( \ell \in \mathbb{N} \) with \( 0 \leq \ell < k \),
\[
   s_{P(\langle a^1, \ldots, a^\ell \rangle)}(\langle a^1, \ldots, a^\ell \rangle) = a^{\ell+1}.
\]

**Example (Outcome)**
\[
   O(AF, C) = \langle A, C, F \rangle \quad \text{and} \quad O(AE, D) = \langle A, D \rangle.
\]
Motivation
Definitions
Solution Concepts
One-Deviation Property
Kuhn’s Theorem
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Summary

3 Solution Concepts

Motivation
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Summary

Nash Equilibria

Definition (Nash equilibrium in an extensive game)
A Nash equilibrium in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is a strategy profile $s^*$ such that for every player $i \in N$ and for all strategies $s_i \in S_i$,

$$u_i(O(s^*_{-i}, s_i)) \geq u_i(O(s^*_{-i}, s'_i)).$$

Induced Strategic Game

Definition (Induced strategic game)
The strategic game $G$ induced by an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is defined by $G = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$, where

- $A'_i = S_i$ for all $i \in N$, and
- $u'_i(a) = u_i(O(a))$ for all $i \in N$.

Proposition
The Nash equilibria of an extensive game $\Gamma$ are exactly the Nash equilibria of the induced strategic game $G$ of $\Gamma$.

Induced Strategic Game

Remarks:
- Each extensive game can be transformed into a strategic game, but the resulting game can be exponentially larger.
- The other direction does not work, because in extensive games, we do not have simultaneous actions.
Empty Threats

Example (Empty threat)

Extensive game:

```
      T  B
     /   |
    /    |
   (0,0) |
    L    |
     1.2 |
    R    |
(2,1)  |
```

Strategic form:

```
      |   | R
---+---+---
L   | 0.0| 2.1
T   | 1.2| 1.2
```

Nash equilibria: \( (B, L) \) and \( (T, R) \).

However, \( (B, L) \) is not realistic:

- Player 1 plays \( B \), "fearing" response \( L \) to \( T \).
- But player 2 would never play \( L \) in the extensive game.

\( (B, L) \) involves "empty threat".

Subgames

Definition (Subgame)

A **subgame** of an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \), starting after history \( h \), is the game \( \Gamma(h) = \langle N, H|_h, P|_h, (u_i|_h)_{i \in N} \rangle \), where

- \( H|_h = \{ h' \mid (h, h') \in H \} \),
- \( P|_h(h') = P(h, h') \) for all \( h' \in H|_h \), and
- \( u_i|_h(h') = u_i(h, h') \) for all \( h' \in H|_h \).

Subgame-Perfect Equilibria

Definition (Subgame-perfect equilibrium)

A strategy profile \( s^* \) in an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \) is a **subgame-perfect equilibrium** if and only if for every player \( i \in N \) and every nonterminal history \( h \in H \setminus \emptyset \) with \( P(h) = i \),

\[
u_i|_h(O_h(s^*_i|h, s^*_i|h)) \geq u_i|_h(O_h(s^*_i|h, s_i))
\]

for every strategy \( s_i \in S_i \) in subgame \( \Gamma(h) \).
Two Nash equilibria:
- \((T, R)\): subgame-perfect, because:
  - In history \(h = (T)\): subgame-perfect.
  - In history \(h = ()\): player 1 obtains utility 1 when choosing \(B\) and utility of 2 when choosing \(T\).
- \((B, L)\): not subgame-perfect, since \(L\) does not maximize the utility of player 2 in history \(h = (T)\).

Example (Subgame-perfect equilibria in division game)

Equilibria in subgames:
- in \(\Gamma((2, 0))\): \(y\) and \(n\)
- in \(\Gamma((1, 1))\): only \(y\)
- in \(\Gamma((0, 2))\): only \(y\)
- in \(\Gamma()\): \((2, 0), yyy\) and \((1, 1), nyy\)

Nash equilibria (red: empty threat):
- \(((2, 0), yyy), (2, 0), yyn\), \(((2, 0), yny), ((2, 0), ynn), (2, 0), nny\), \(((2, 0), yyn), (2, 0), nnn\), \(((1, 1), nyy), (1, 1), nyn\), \(((0, 2), nny), (0, 2), nnn\).
Motivation

Positive case (a subgame-perfect equilibrium exists):

- **Step 1**: Show that it suffices to consider local deviations from strategies (for finite-horizon games).
- **Step 2**: Show how to systematically explore such local deviations to find a subgame-perfect equilibrium (for finite games).

Step 1: One-Deviation Property

Definition (One-deviation property)

A strategy profile \( s^* \) in an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \) satisfies the one-deviation property if and only if for every player \( i \in N \) and every nonterminal history \( h \) with \( P(h) = i \),

\[
 u_i|_h(O_h(s^*_i|h, s_i)) \geq u_i|_h(O_h(s^*_i|h, s_i))
\]

for every strategy \( s_i \in S_i \) in subgame \( \Gamma(h) \) that differs from \( s^*_i|h \) only in the action it prescribes after the initial history of \( \Gamma(h) \).

**Note**: Without the highlighted parts, this is just the definition of subgame-perfect equilibria!
Step 1: One-Deviation Property

Proof (ctd.)

$(\Leftarrow)$ ...WLOG, the number of histories $h'$ with $s_i(h') \neq s_i^*|_{h(h')} \text{ is at most } \ell(\Gamma(h)) \text{ and hence finite (finite horizon assumption!)}$, since deviations not on resulting outcome path are irrelevant.

Illustration: strategies $s_1^*|_h = AGILN$ and $s_2^*|_h = CF$ red:

\[ P(h) = 1 \]

Then only $B$ and $O$ really matter.

Step 1: One-Deviation Property

Proof (ctd.)

$(\Leftarrow)$ ... Illustration for WLOG assumption: Assume $s_1 = BHKMO$ (blue) profitable deviation:

\[ P(h) = 1 \]

Choose profitable deviation $s_i \in \Gamma(h)$ with minimal number of deviation points (such $s_i$ must exist).

Let $h^*$ be the longest history in $\Gamma(h)$ with $s_i(h^*) \neq s_i^*|_{h(h^*)}$, i.e., “deepest” deviation point for $s_i$.

Then in $\Gamma(h, h^*)$, $s_i|_{h^*}$ differs from $s_i^*|_{(h, h^*)}$ only in the initial history.

Moreover, $s_i|_{h^*}$ is a profitable deviation in $\Gamma(h, h^*)$, since $h^*$ is the longest history in $\Gamma(h)$ with $s_i(h^*) \neq s_i^*|_{h(h^*)}$.

So, $\Gamma(h, h^*)$ is the desired subgame where a one-step deviation is sufficient to improve utility.
Step 1: One-Deviation Property

Example

\[ \text{To show that } (AHI, CE) \text{ is a subgame-perfect equilibrium, it suffices to check these deviating strategies:} \]

Player 1:  
- G in subgame $\Gamma(\langle A, C \rangle)$
- K in subgame $\Gamma(\langle B, F \rangle)$
- BHI in $\Gamma$

Player 2:  
- D in subgame $\Gamma(\langle A \rangle)$
- F in subgame $\Gamma(\langle B \rangle)$

In particular, e.g., no need to check if strategy BGK of player 1 is profitable in $\Gamma$.

Step 2: Kuhn’s Theorem

Theorem (Kuhn)  
Every finite extensive game has a subgame-perfect equilibrium.

Proof idea:  
- Proof is constructive and builds a subgame-perfect equilibrium bottom-up (aka backward induction).
- For those familiar with the Foundations of AI lecture: generalization of Minimax algorithm to general-sum games with possibly more than two players.
Step 2: Kuhn’s Theorem

A bit more formally:

**Proof**

Let $\Gamma = (N, H, P, (u_i)_{i \in N})$ be a finite extensive game. Construct a subgame-perfect equilibrium by induction on $\ell(\Gamma(h))$ for all subgames $\Gamma(h)$. In parallel, construct functions $t_i : H \to \mathbb{R}$ for all players $i \in N$ s.t. $t_i(h)$ is the payoff for player $i$ in a subgame-perfect equilibrium in subgame $\Gamma(h)$.

**Base case:** If $\ell(\Gamma(h)) = 0$, then $t_i(h) = u_i(h)$ for all $i \in N$.

...
Step 2: Kuhn’s Theorem
Remark on Infinite Games

Corresponding proposition for infinite games does not hold.

Counterexamples (both for one-player case):

A) finite horizon, infinite branching factor:
Infinitely many actions \( a \in A = [0,1) \) with payoffs
\( u_1(\langle a \rangle) = a \) for all \( a \in A \).
There exists no subgame-perfect equilibrium in this game.

\( u_1(CCC\ldots) = 0 \) and \( u_1(CC\ldots CS) = n + 1. \)
No subgame-perfect equilibrium.

Uniqueness:
Kuhn’s theorem tells us nothing about uniqueness of subgame-perfect equilibria. However, if no two histories get the same evaluation by any player, then the subgame-perfect equilibrium is unique.

6 Two Extensions

- Chance
- Simultaneous Moves
**Chance Moves**

### Definition

An extensive game with chance moves is a tuple \( \Gamma = (N, H, P, f_c, \{u_i\}_{i \in N}) \), where

- \( N, A, H \) and \( u_i \) are defined as before,
- the player function \( P : H \setminus Z \rightarrow N \cup \{c\} \) can also take the value \( c \) for a chance node, and
- for each \( h \in H \setminus Z \) with \( P(h) = c \), the function \( f_c(\cdot | h) \) is a probability distribution on \( A(h) \) such that the probability distributions for all \( h \in H \) are independent of each other.

### Example

\[
\begin{align*}
P(\langle \rangle) &= 1, \\
P(\langle A \rangle) &= c, \\
P(\langle B \rangle) &= c, \\
P(\langle B, F \rangle) &= \frac{2}{3}, \\
P(\langle B, G \rangle) &= 2, \\
P(\langle F \rangle) &= \frac{1}{2}, \\
P(\langle G \rangle) &= \frac{1}{3}.
\end{align*}
\]

### Remark:

The one-deviation property and Kuhn’s theorem still hold in the presence of chance moves. When proving Kuhn’s theorem, expected utilities have to be used.
Simultaneous Moves

Definition
An extensive game with simultaneous moves is a tuple
\( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \), where
- \( N, A, H \) and \((u_i)\) are defined as before, and
- \( P : H \rightarrow 2^N \) assigns to each nonterminal history a set of players to move; for all \( h \in H \setminus Z \), there exists a family \((A_i(h))_{i \in P(h)}\) such that
  \[
  A(h) = \{ (h, a) \in H \} = \prod_{i \in P(h)} A_i(h).
  \]

Simultaneous Moves

One-Deviation Property and Kuhn’s Theorem

Remark:
- The one-deviation property still holds for extensive game with perfect information and simultaneous moves.
- Kuhn’s theorem does not hold for extensive game with simultaneous moves.

Example: Matching Pennies can be viewed as extensive game with simultaneous moves. No Nash equilibrium/subgame-perfect equilibrium.

<table>
<thead>
<tr>
<th></th>
<th>( H )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>player 2</td>
<td>1,−1</td>
<td>−1, 1</td>
</tr>
<tr>
<td>player 1</td>
<td>( H )</td>
<td>( T )</td>
</tr>
<tr>
<td></td>
<td>−1, 1</td>
<td>1, −1</td>
</tr>
</tbody>
</table>

\[ \Rightarrow \] Need more sophisticated solution concepts (cf. mixed strategies). Not covered in this lecture.

Simultaneous Moves

Intended meaning of simultaneous moves: All players from \( P(h) \) move simultaneously.

Strategies: Functions \( s_i : h \mapsto a_i \) with \( a_i \in A_i(h) \).

Histories: Sequences of vectors of actions.

Outcome: Terminal history reached when tracing strategy profile.

Payoffs: Utilities at outcome history.

Simultaneous Moves

Example: Three-Person Cake Splitting Game

Setting:
- Three players have to split a cake fairly.
- Player 1 suggest split: shares \( x_1, x_2, x_3 \in [0, 1] \) s.t. \( x_1 + x_2 + x_3 = 1 \).
- Then players 2 and 3 simultaneously and independently decide whether to accept (“y”) or reject (“n”) the suggested splitting.
- If both accept, each player \( i \) gets his allotted share (utility \( x_i \)). Otherwise, no player gets anything (utility 0).

Simultaneous Moves
Simultaneous Moves
Example: Three-Person Cake Splitting Game

Formally:

\[ N = \{1, 2, 3\} \]
\[ X = \{ (x_1, x_2, x_3) \in [0, 1]^3 \mid x_1 + x_2 + x_3 = 1 \} \]
\[ H = \{ () \} \cup \{ (x) \mid x \in X \} \cup \{ (x, z) \mid x \in X, z \in \{y, n\} \times \{y, n\} \} \]
\[ P((\)) = \{ \{\} \} \]
\[ P((x)) = \{ 2, 3 \} \text{ for all } x \in X \]
\[ u_i((x, z)) = \begin{cases} 0 & \text{if } z \in \{(y, n), (n, y), (n, n)\} \\ x_i & \text{if } z = (y, y). \end{cases} \text{ for all } i \in N \]

Simultaneous Moves
Example: Three-Person Cake Splitting Game

Subgame-perfect equilibria:

- **Entire game:**
  - Let \( s_2 \) and \( s_3 \) be any two strategies of players 2 and 3 such that for all splits \( x \in X \) the profile \((s_2((x)), s_3((x)))\) is one of the NEs from above.
  - Let \( X_y = \{ x \in X \mid s_2((x)) = s_3((x)) = y \} \) be the set of splits accepted under \( s_2 \) and \( s_3 \). Distinguish three cases:
    - \( X_y = \emptyset \) or \( x_1 = 0 \) for all \( x \in X_y \). Then \((s_1, s_2, s_3)\) is a subgame-perfect equilibrium for any possible \( s_1 \).
    - \( X_y \neq \emptyset \) and there are splits \( x_{max} = (x_1, x_2, x_3) \in X_y \) that maximize \( x_1 > 0 \). Then \((s_1, s_2, s_3)\) is a subgame-perfect equilibrium if and only if \( s_1((\)) \) is such a split \( x_{max} \).
    - \( X_y \neq \emptyset \) and there are no splits \( (x_1, x_2, x_3) \in X_y \) that maximize \( x_1 \). Then there is no subgame-perfect equilibrium, in which player 2 follows strategy \( s_2 \) and player 3 follows strategy \( s_3 \).
For finite-horizon extensive games, it suffices to consider local deviations when looking for better strategies.

For infinite-horizon games, this is not true in general.

Every finite extensive game has a subgame-perfect equilibrium.

This does not generally hold for infinite games, no matter is game is infinite due to infinite branching factor or infinitely long histories (or both).

With chance moves, one deviation property and Kuhn’s theorem still hold.

With simultaneous moves, Kuhn’s theorem no longer holds.