Game Theory
6. Extensive Games

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May 31st, 2016
1 Motivation
So far: All players move \textit{simultaneously}, and then the outcome is determined.

Often in practice: Several moves in \textit{sequence} (e.g. in chess).

\(\sim\) cannot be directly reflected by strategic games.

Extensive games (with perfect information) reflect such situations by modeling games as \textit{game trees}.

Idea: Players have several decision points where they can decide how to play.

Strategies: Mappings from decision points in the game tree to actions to be played.
2 Definitions
**Definition (Extensive game with perfect information)**

An extensive game with perfect information is a tuple \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \) that consists of:

- A finite non-empty set \( N \) of players.
- A set \( H \) of (finite or infinite) sequences, called histories, such that
  - the empty sequence \( \langle \rangle \in H \),
  - \( H \) is closed under prefixes: if \( \langle a^1, \ldots, a^k \rangle \in H \) for some \( k \in \mathbb{N} \cup \{ \infty \} \), and \( l < k \), then also \( \langle a^1, \ldots, a^l \rangle \in H \), and
  - \( H \) is closed under limits: if for some infinite sequence \( \langle a^i \rangle_{i=1}^\infty \), we have \( \langle a^i \rangle_{i=1}^k \in H \) for all \( k \in \mathbb{N} \), then \( \langle a^i \rangle_{i=1}^\infty \in H \).

All infinite histories and all histories \( \langle a^i \rangle_{i=1}^k \in H \), for which there is no \( a^{k+1} \) such that \( \langle a^i \rangle_{i=1}^{k+1} \in H \) are called terminal histories \( Z \). Components of a history are called actions.
Definition (Extensive game with perfect information, ctd.)

- A player function $P : H \setminus Z \rightarrow N$ that determines which player’s turn it is to move after a given nonterminal history.
- For each player $i \in N$, a utility function (or payoff function) $u_i : Z \rightarrow \mathbb{R}$ defined on the set of terminal histories.

The game is called **finite**, if $H$ is finite. It has a **finite horizon**, if the length of histories is bounded from above.

**Assumption:** All ingredients of $\Gamma$ are common knowledge amongst the players of the game.

**Terminology:** In the following, we will simply write extensive games instead of extensive games with perfect information.
Example (Division game)

- Two identical objects should be divided among two players.
- Player 1 proposes an allocation.
- Player 2 agrees or rejects.
  - On agreement: Allocation as proposed.
  - On rejection: Nobody gets anything.
Extensive Games

Example (Division game, formally)

\[ N = \{1, 2\} \]

\[ H = \{\langle\rangle, \langle(2, 0)\rangle, \langle(1, 1)\rangle, \langle(0, 2)\rangle, \langle(2, 0), y\rangle, \langle(2, 0), n\rangle, \ldots\} \]

\[ P(\langle\rangle) = 1, \ P(h) = 2 \text{ for all } h \in H \setminus Z \text{ with } h \neq \langle\rangle \]

\[ u_1(\langle(2, 0), y\rangle) = 2, \ u_2(\langle(2, 0), y\rangle) = 0, \text{ etc.} \]
Notation:

Let $h = \langle a^1, \ldots, a^k \rangle$ be a history, and $a$ an action.

- Then $(h, a)$ is the history $\langle a^1, \ldots, a^k, a \rangle$.
- If $h' = \langle b^1, \ldots, b^\ell \rangle$, then $(h, h')$ is the history $\langle a^1, \ldots, a^k, b^1, \ldots, b^\ell \rangle$.
- The set of actions from which player $P(h)$ can choose after a history $h \in H \setminus Z$ is written as

$$A(h) = \{a \mid (h, a) \in H\}.$$
Definition (Strategy in an extensive game)

A strategy of a player $i$ in an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \) is a function \( s_i \) that assigns to each nonterminal history \( h \in H \setminus Z \) with \( P(h) = i \) an action \( a \in A(h) \). The set of strategies of player $i$ is denoted as \( S_i \).

Remark: Strategies require us to assign actions to histories \( h \), even if it is clear that they will never be played (e.g., because \( h \) will never be reached because of some earlier action).

Notation (for finite games): A strategy for a player is written as a string of actions at decision nodes as visited in a breadth-first order.
Example (Strategies in an extensive game)

- **Strategies for player 1:** $AE$, $AF$, $BE$ and $BF$
- **Strategies for player 2:** $C$ and $D$. 
**Definition (Outcome)**

The outcome $O(s)$ of a strategy profile $s = (s_i)_{i \in N}$ is the (possibly infinite) terminal history $h = \langle a^i \rangle_{i=1}^k$, with $k \in \mathbb{N} \cup \{\infty\}$, such that for all $\ell \in \mathbb{N}$ with $0 \leq \ell < k$,

$$S_P(\langle a^1, \ldots, a^\ell \rangle)(\langle a^1, \ldots, a^\ell \rangle) = a^{\ell+1}.$$  

**Example (Outcome)**

![Game Tree Diagram](image)

- $P(\langle \rangle) = 1$
- $P(\langle A \rangle) = 2$
- $P(\langle A, C \rangle) = 1$
- $P(\langle A, D \rangle) = 1$
- $P(\langle B \rangle) = 1$  

Given the outcomes:

- $O(AF, C) = \langle A, C, F \rangle$
- $O(AE, D) = \langle A, D \rangle$.  

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3 Solution Concepts
Definition (Nash equilibrium in an extensive game)

A Nash equilibrium in an extensive game \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \) is a strategy profile \( s^* \) such that for every player \( i \in N \) and for all strategies \( s_i \in S_i \),

\[
u_i(O(s_{-i}^*, s_i^*)) \geq u_i(O(s_{-i}^*, s_i)).\]
Induced Strategic Game

Definition (Induced strategic game)

The strategic game $G$ induced by an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is defined by $G = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$, where

- $A'_i = S_i$ for all $i \in N$, and
- $u'_i(a) = u_i(O(a))$ for all $i \in N$.

Proposition

The Nash equilibria of an extensive game $\Gamma$ are exactly the Nash equilibria of the induced strategic game $G$ of $\Gamma$.  

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Remarks:

- Each extensive game can be transformed into a strategic game, but the resulting game can be exponentially larger.
- The other direction does not work, because in extensive games, we do not have simultaneous actions.
Empty Threats

Example (Empty threat)

Extensive game:

\[
\begin{array}{c}
T \\
B
\end{array}
\]

\[
\begin{array}{c}
L \\
R
\end{array}
\]

\[
\begin{array}{c}
P(\langle \rangle) = 1 \\
P(\langle T \rangle) = 2
\end{array}
\]

Strategic form:

\[
\begin{array}{cc|cc}
& L & R \\
T & 0,0 & 2,1 \\
B & 1,2 & 1,2
\end{array}
\]

Nash equilibria: \((B, L)\) and \((T, R)\).

However, \((B, L)\) is not realistic:

- Player 1 plays \(B\), “fearing” response \(L\) to \(T\).
- But player 2 would never play \(L\) in the extensive game.
  \(\rightarrow\) \((B, L)\) involves “empty threat”.

Strategies:

- Player 1: \(T\) and \(B\)
- Player 2: \(L\) and \(R\)
Motivation
Definitions
Solution
Concepts
One-Deviation Property
Kuhn's Theorem
Two Extensions
Summary

Subgames

Idea: Exclude empty threats.

How? Demand that a strategy profile is not only a Nash equilibrium in the strategic form, but also in every subgame.

Definition (Subgame)

A subgame of an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$, starting after history $h$, is the game $\Gamma(h) = \langle N, H|_h, P|_h, (u_i|_h)_{i \in N} \rangle$, where

- $H|_h = \{ h' \mid (h, h') \in H \}$,
- $P|_h(h') = P(h, h')$ for all $h' \in H|_h$, and
- $u_i|_h(h') = u_i(h, h')$ for all $h' \in H|_h$. 
Definition (Strategy in a subgame)
Let $\Gamma$ be an extensive game and $\Gamma(h)$ a subgame of $\Gamma$ starting after some history $h$. For each strategy $s_i$ of $\Gamma$, let $s_i|_h$ be the strategy induced by $s_i$ for $\Gamma(h)$. Formally, for all $h' \in H|_h$,

$$s_i|_h(h') = s_i(h, h').$$

The outcome function of $\Gamma(h)$ is denoted by $O_h$. 
Definition (Subgame-perfect equilibrium)

A strategy profile $s^*$ in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is a subgame-perfect equilibrium if and only if for every player $i \in N$ and every nonterminal history $h \in H \setminus Z$ with $P(h) = i$,

$$u_i|_h(O_h(s^{*-i}|_h, s^*|_h)) \geq u_i|_h(O_h(s^{*-i}|_h, s_i))$$

for every strategy $s_i \in S_i$ in subgame $\Gamma(h)$. 

Subgame-Perfect Equilibria
Two Nash equilibria:

- \((T, R)\): subgame-perfect, because:
  - In history \(h = \langle T \rangle\): subgame-perfect.
  - In history \(h = \langle \rangle\): player 1 obtains utility 1 when choosing \(B\) and utility of 2 when choosing \(T\).

- \((B, L)\): not subgame-perfect, since \(L\) does not maximize the utility of player 2 in history \(h = \langle T \rangle\).
Subgame-Perfect Equilibria

Example (Subgame-perfect equilibria in division game)

Equilibria in subgames:
- in $\Gamma(\langle 2, 0 \rangle)$: y and n
- in $\Gamma(\langle 1, 1 \rangle)$: only y
- in $\Gamma(\langle 0, 2 \rangle)$: only y
- in $\Gamma(\langle \rangle)$: $((2, 0), yyy)$ and $((1, 1), nyy)$

Nash equilibria (red: empty threat):
- $((2, 0), yyy)$, $((2, 0), yyn)$, $((2, 0), yny)$, $((2, 0), ynn)$, $((2, 0), nny)$, $((2, 0), nnn)$,
- $((1, 1), nyy)$, $((1, 1), nyn)$,
- $((0, 2), nny)$, $((0, 2), nnn)$. 
4 One-Deviation Property
Motivation

■ Existence:
  ■ Does every extensive game have a subgame-perfect equilibrium?
  ■ If not, which extensive games do have a subgame-perfect equilibrium?

■ Computation:
  ■ If a subgame-perfect equilibrium exists, how to compute it?
  ■ How complex is that computation?
Positive case (a subgame-perfect equilibrium exists):

- **Step 1**: Show that it suffices to consider local deviations from strategies (for finite-horizon games).

- **Step 2**: Show how to systematically explore such local deviations to find a subgame-perfect equilibrium (for finite games).
Step 1: One-Deviation Property

Definition

Let $\Gamma$ be a finite-horizon extensive game. Then $\ell(\Gamma)$ denotes the length of the longest history of $\Gamma$. 
Step 1: One-Deviation Property

Definition (One-deviation property)

A strategy profile $s^*$ in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N}\rangle$ satisfies the one-deviation property if and only if for every player $i \in N$ and every nonterminal history $h \in H \setminus Z$ with $P(h) = i$,

$$u_i|_h(O_h(s^*_{-i}|h, s^*_i|h)) \geq u_i|_h(O_h(s^*_{-i}|h, s_i))$$

for every strategy $s_i \in S_i$ in subgame $\Gamma(h)$ that differs from $s^*_i|h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

Note: Without the highlighted parts, this is just the definition of subgame-perfect equilibria!
Step 1: One-Deviation Property

Lemma
Let $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ be a finite-horizon extensive game. Then a strategy profile $s^*$ is a subgame-perfect equilibrium of $\Gamma$ if and only if it satisfies the one-deviation property.

Proof
- $(\Rightarrow)$ Clear.
- $(\Leftarrow)$ By contradiction:
  Suppose that $s^*$ is not a subgame-perfect equilibrium. Then there is a history $h$ and a player $i$ such that $s_i$ is a profitable deviation for player $i$ in subgame $\Gamma(h)$.

...
Step 1: One-Deviation Property

Proof (ctd.)

\(\leftarrow\) ... WLOG, the number of histories \(h'\) with 
\(s_i(h') \neq s_i^*|_{h(h')}\) is at most \(\ell(\Gamma(h))\) and hence finite (finite 
horizon assumption!), since deviations not on resulting 
outcome path are irrelevant.

Illustration: strategies \(s_1^*|_{h} = AGILN\) and \(s_2^*|_{h} = CF\) red:
Step 1: One-Deviation Property

Proof (ctd.)

\[ (\Leftarrow) \text{ ... Illustration for WLOG assumption: Assume } s_1 = BHKMO \text{ (blue) profitable deviation:} \]

\[ P(h) = 1 \]

Then only B and O really matter.
Step 1: One-Deviation Property

Proof (ctd.)

- (⇐) … Illustration for WLOG assumption: And hence

\( \tilde{s}_1 = BG\overline{ILO} \) (blue) also profitable deviation:

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Step 1: One-Deviation Property

Proof (ctd.)

$(\Leftarrow) \ldots$

Choose profitable deviation $s_i$ in $\Gamma(h)$ with minimal number of deviation points (such $s_i$ must exist).

Let $h^*$ be the longest history in $\Gamma(h)$ with $s_i(h^*) \neq s_i^*|_h(h^*)$, i.e., “deepest” deviation point for $s_i$.

Then in $\Gamma(h, h^*)$, $s_i|_{h^*}$ differs from $s_i^*|_{(h, h^*)}$ only in the initial history.

Moreover, $s_i|_{h^*}$ is a profitable deviation in $\Gamma(h, h^*)$, since $h^*$ is the longest history in $\Gamma(h)$ with $s_i(h^*) \neq s_i^*|_h(h^*)$.

So, $\Gamma(h, h^*)$ is the desired subgame where a one-step deviation is sufficient to improve utility.
Step 1: One-Deviation Property

Example

To show that \((AHI, CE)\) is a subgame-perfect equilibrium, it suffices to check these deviating strategies:

**Player 1:**
- \(G\) in subgame \(\Gamma(\langle A, C \rangle)\)
- \(K\) in subgame \(\Gamma(\langle B, F \rangle)\)
- \(BHI\) in \(\Gamma\)

**Player 2:**
- \(D\) in subgame \(\Gamma(\langle A \rangle)\)
- \(F\) in subgame \(\Gamma(\langle B \rangle)\)

In particular, e.g., no need to check if strategy \(BGK\) of player 1 is profitable in \(\Gamma\).
The corresponding proposition for infinite-horizon games does not hold.

Counterexample (one-player case):

Strategy \( s_i \) with \( s_i(h) = S \) for all \( h \in H \setminus Z \)

- satisfies one deviation property, but
- is not a subgame-perfect equilibrium, since it is dominated by \( s_i^* \) with \( s_i^*(h) = C \) for all \( h \in H \setminus Z \).
5 Kuhn’s Theorem
Step 2: Kuhn’s Theorem

Theorem (Kuhn)
Every finite extensive game has a subgame-perfect equilibrium.

Proof idea:
- Proof is constructive and builds a subgame-perfect equilibrium bottom-up (aka backward induction).
- For those familiar with the Foundations of AI lecture: generalization of Minimax algorithm to general-sum games with possibly more than two players.
Step 2: Kuhn’s Theorem

Example

\[
\begin{align*}
 s_2(\langle A \rangle) &= C & t_1(\langle A \rangle) &= 1 & t_2(\langle A \rangle) &= 5 \\
 s_2(\langle B \rangle) &= F & t_1(\langle B \rangle) &= 0 & t_2(\langle B \rangle) &= 8 \\
 s_1(\langle \rangle) &= A & t_1(\langle \rangle) &= 1 & t_2(\langle \rangle) &= 5
\end{align*}
\]
Step 2: Kuhn’s Theorem

A bit more formally:

Proof

Let $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ be a finite extensive game. Construct a subgame-perfect equilibrium by induction on $\ell(\Gamma(h))$ for all subgames $\Gamma(h)$. In parallel, construct functions $t_i : H \rightarrow \mathbb{R}$ for all players $i \in N$ s. t. $t_i(h)$ is the payoff for player $i$ in a subgame-perfect equilibrium in subgame $\Gamma(h)$.

Base case: If $\ell(\Gamma(h)) = 0$, then $t_i(h) = u_i(h)$ for all $i \in N$.

\ldots
Step 2: Kuhn’s Theorem

Proof (ctd.)

**Inductive case:** If $t_i(h)$ already defined for all $h \in H$ with $\ell(\Gamma(h)) \leq k$, consider $h^* \in H$ with $\ell(\Gamma(h^*)) = k + 1$ and $P(h^*) = i$. For all $a \in A(h^*)$, $\ell(\Gamma(h^*, a)) \leq k$, let

$$s_i(h^*) := \arg\max_{a \in A(h^*)} t_i(h^*, a) \quad \text{and} \quad t_j(h^*) := t_j(h^*, s_i(h^*)) \quad \text{for all players } j \in N.$$ 

Inductively, we obtain a strategy profile $s$ that satisfies the one-deviation property.

With the one-deviation property lemma it follows that $s$ is a subgame-perfect equilibrium. \qed
Step 2: Kuhn’s Theorem

- In principle: sample subgame-perfect equilibrium effectively computable using the technique from the above proof.
- In practice: often game trees not enumerated in advance, hence unavailable for backward induction.
- E.g., for branching factor $b$ and depth $m$, procedure needs time $O(b^m)$. 
Corresponding proposition for infinite games does not hold.

Counterexamples (both for one-player case):

A) finite horizon, infinite branching factor:

Infinitely many actions \( a \in A = [0, 1) \) with payoffs \( u_1(\langle a \rangle) = a \) for all \( a \in A \).

There exists no subgame-perfect equilibrium in this game.
Step 2: Kuhn’s Theorem
Remark on Infinite Games

B) infinite horizon, finite branching factor:

\[
\begin{align*}
S & \quad C & \quad C & \quad C & \quad C & \quad C & \quad C, \ldots \\
1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 0
\end{align*}
\]

\[u_1(CCC\ldots) = 0 \text{ and } u_1(\underbrace{CC\ldots CS}_n) = n + 1.\]

No subgame-perfect equilibrium.
Step 2: Kuhn’s Theorem

Uniqueness:

Kuhn’s theorem tells us nothing about uniqueness of subgame-perfect equilibria. However, if no two histories get the same evaluation by any player, then the subgame-perfect equilibrium is unique.
6 Two Extensions

- Chance
- Simultaneous Moves
**Chance Moves**

**Definition**

An extensive game with chance moves is a tuple $\Gamma = \langle N, H, P, f_c, (u_i)_{i \in N} \rangle$, where

- $N$, $A$, $H$ and $u_i$ are defined as before,
- the player function $P : H \setminus Z \rightarrow N \cup \{c\}$ can also take the value $c$ for a chance node, and
- for each $h \in H \setminus Z$ with $P(h) = c$, the function $f_c(\cdot | h)$ is a probability distribution on $A(h)$ such that the probability distributions for all $h \in H$ are independent of each other.
Intended meaning of chance moves: In chance node, an applicable action is chosen randomly with probability according to $f_c$.

Strategies: Defined as before.

Outcome: For a given strategy profile, the outcome is a probability distribution on the set of terminal histories.

Payoffs: For player $i$, $U_i$ is the expected payoff (with weights according to outcome probabilities).
Chance Moves

Example

\[
P(\langle A \rangle) = c \quad P(\langle B \rangle) = c
\]

\[
f_c(D|\langle A \rangle) = \frac{1}{2} \quad f_c(E|\langle A \rangle) = \frac{1}{2} \quad f_c(F|\langle B \rangle) = \frac{1}{3} \quad f_c(G|\langle B \rangle) = \frac{2}{3}
\]

\[
P(\langle B, F \rangle) = 2 \quad P(\langle B, G \rangle) = 2
\]

\[
(0, 6) \quad (2, 2) \quad (0, 3) \quad (3, 3) \quad (2, 2) \quad (3, 3)
\]
Remark:
The one-deviation property and Kuhn’s theorem still hold in the presence of chance moves. When proving Kuhn’s theorem, expected utilities have to be used.
Simultaneous Moves

Definition

An **extensive game with simultaneous moves** is a tuple \( \Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle \), where

- \( N, A, H \) and \( (u_i) \) are defined as before, and
- \( P : H \to 2^N \) assigns to each nonterminal history a set of players to move; for all \( h \in H \setminus Z \), there exists a family \( (A_i(h))_{i \in P(h)} \) such that

\[
A(h) = \{ a \mid (h, a) \in H \} = \prod_{i \in P(h)} A_i(h).
\]
Simultaneous Moves

- **Intended meaning of simultaneous moves**: All players from $P(h)$ move simultaneously.
- **Strategies**: Functions $s_i : h \mapsto a_i$ with $a_i \in A_i(h)$.
- **Histories**: Sequences of vectors of actions.
- **Outcome**: Terminal history reached when tracing strategy profile.
- **Payoffs**: Utilities at outcome history.
Remark:

- The **one-deviation property still holds** for extensive game with perfect information and simultaneous moves.
- **Kuhn’s theorem does not hold** for extensive game with simultaneous moves.

**Example:** Matching Pennies can be viewed as extensive game with simultaneous moves. No Nash equilibrium/subgame-perfect equilibrium.

\[
\begin{array}{c|cc}
& H & T \\
\hline
H & 1, -1 & -1, 1 \\
T & -1, 1 & 1, -1 \\
\end{array}
\]

\[\Rightarrow\] Need more sophisticated solution concepts (cf. mixed strategies). Not covered in this lecture.
Simultaneous Moves
Example: Three-Person Cake Splitting Game

Setting:

- Three players have to split a cake fairly.
- Player 1 suggest split: shares $x_1, x_2, x_3 \in [0, 1]$ s.t. $x_1 + x_2 + x_3 = 1$.
- Then players 2 and 3 simultaneously and independently decide whether to accept (“y”) or reject (“n”) the suggested splitting.
- If both accept, each player $i$ gets his allotted share (utility $x_i$). Otherwise, no player gets anything (utility 0).
Simultaneous Moves
Example: Three-Person Cake Splitting Game

Formally:

\[
N = \{1, 2, 3\} \\
X = \{(x_1, x_2, x_3) \in [0, 1]^3 \mid x_1 + x_2 + x_3 = 1\} \\
H = \{\langle\rangle\} \cup \{\langle x \rangle \mid x \in X\} \cup \{\langle x, z \rangle \mid x \in X, z \in \{y, n\} \times \{y, n\}\} \\
P(\langle\rangle) = \{1\} \\
P(\langle x \rangle) = \{2, 3\} \text{ for all } x \in X \\
u_i(\langle x, z \rangle) = \begin{cases} 
0 & \text{if } z \in \{(y, n), (n, y), (n, n)\} \\
x_i & \text{if } z = (y, y).
\end{cases} \text{ for all } i \in N
\]
Simultaneous Moves
Example: Three-Person Cake Splitting Game

Subgame-perfect equilibria:

- Subgames after legal split \((x_1, x_2, x_3)\) by player 1:
  - NE \((y, y)\) (both accept)
  - NE \((n, n)\) (neither accepts)
  - If \(x_2 = 0\), NE \((n, y)\) (only player 3 accepts)
  - If \(x_3 = 0\), NE \((y, n)\) (only player 2 accepts)
Simultaneous Moves
Example: Three-Person Cake Splitting Game

Subgame-perfect equilibria (ctd.):

**Entire game:**
Let $s_2$ and $s_3$ be any two strategies of players 2 and 3 such that for all splits $x \in X$ the profile $(s_2(\langle x \rangle), s_3(\langle x \rangle))$ is one of the NEs from above.

Let $X_y = \{x \in X \mid s_2(\langle x \rangle) = s_3(\langle x \rangle) = y\}$ be the set of splits accepted under $s_2$ and $s_3$. Distinguish three cases:

- $X_y = \emptyset$ or $x_1 = 0$ for all $x \in X_y$. Then $(s_1, s_2, s_3)$ is a subgame-perfect equilibrium for any possible $s_1$.
- $X_y \neq \emptyset$ and there are splits $x_{\max} = (x_1, x_2, x_3) \in X_y$ that maximize $x_1 > 0$. Then $(s_1, s_2, s_3)$ is a subgame-perfect equilibrium if and only if $s_1(\langle \rangle)$ is such a split $x_{\max}$.
- $X_y \neq \emptyset$ and there are no splits $(x_1, x_2, x_3) \in X_y$ that maximize $x_1$. Then there is no subgame-perfect equilibrium, in which player 2 follows strategy $s_2$ and player 3 follows strategy $s_3$. 
7 Summary
Summary

- For **finite-horizon extensive games**, it suffices to consider **local deviations** when looking for better strategies.
- For infinite-horizon games, this is not true in general.
- Every **finite extensive game** has a **subgame-perfect equilibrium**.
- This does not generally hold for infinite games, no matter is game is infinite due to infinite branching factor or infinitely long histories (or both).

- With **chance moves**, one deviation property and Kuhn’s theorem still hold.
- With **simultaneous moves**, Kuhn’s theorem no longer holds.