1 Mixed Strategies

Observation: Not every strategic game has a pure-strategy Nash equilibrium (e.g. matching pennies).

Question:
- Can we do anything about that?
- Which strategy to play then?

Idea: Consider randomized strategies.
Mixed Strategies

A mixed strategy is a strategy where a player is allowed to randomize his actions.

Definition (Mixed strategy)
Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic game.

A mixed strategy of player $i$ in $G$ is a probability distribution $\alpha_i \in \Delta(A_i)$ over player $i$'s actions.

For $a_i \in A_i$, $\alpha_i(a_i)$ is the probability for playing $a_i$.

Terminology: When we talk about strategies in $A_i$ specifically, to distinguish them from mixed strategies, we sometimes also call them pure strategies.

Notation
Since each pure strategy $a_i \in A_i$ is equivalent to its induced mixed strategy $\hat{a}_i$,

\[
\hat{a}_i(a'_i) = \begin{cases} 
1 & \text{if } a'_i = a_i \\
0 & \text{otherwise,} 
\end{cases}
\]

we sometimes abuse notation and write $a_i$ instead of $\hat{a}_i$.

Example (Mixed strategies for matching pennies)

\[
\begin{array}{c|cc}
& H & T \\
\hline
H & 1, -1 & -1, 1 \\
T & -1, 1 & 1, -1 \\
\end{array}
\]

$\alpha = (\alpha_1, \alpha_2)$, $\alpha_1(H) = 2/3$, $\alpha_1(T) = 1/3$, $\alpha_2(H) = 1/3$, $\alpha_2(T) = 2/3$.

This induces a probability distribution over $\{H, T\} \times \{H, T\}$:

$\rho_\alpha(H, H) = \alpha_1(H) \cdot \alpha_2(H) = 2/9$, $\quad u_1(H, H) = +1$,
$\rho_\alpha(H, T) = \alpha_1(H) \cdot \alpha_2(T) = 4/9$, $\quad u_1(H, T) = -1$,
$\rho_\alpha(T, H) = \alpha_1(T) \cdot \alpha_2(H) = 1/9$, $\quad u_1(T, H) = -1$,
$\rho_\alpha(T, T) = \alpha_1(T) \cdot \alpha_2(T) = 2/9$, $\quad u_1(T, T) = +1$. 
### Expected Utility

**Definition (Expected utility)**

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile.

The expected utility of $\alpha$ for player $i$ is

$$U_i(\alpha) = \sum_{a \in A_i} p_\alpha(a) u_i(a) = \sum_{a \in A_i} \left( \prod_{j \in N} \alpha_j(a_j) \right) u_i(a).$$

**Example (Mixed strategies for matching pennies (ctd.))**

The expected utilities for player 1 and player 2 are

$$U_1(\alpha_1, \alpha_2) = -\frac{1}{9} \quad \text{and} \quad U_2(\alpha_1, \alpha_2) = \frac{1}{9}.$$

### Mixed Extension

**Definition (Mixed extension)**

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

The mixed extension $G$ is the game $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$ where

- $\Delta(A_i)$ is the set of probability distributions over $A_i$ and
- $U_i : \prod_{i \in N} \Delta(A_i) \rightarrow \mathbb{R}$ assigns to each mixed strategy profile $\alpha$ the expected utility for player $i$ according to the induced probability distribution $p_\alpha$.

### Nash Equilibria in Mixed Strategies

**Definition (Nash equilibrium in mixed strategies)**

Let $G$ be a strategic game.

A Nash equilibrium in mixed strategies (or mixed-strategy Nash equilibrium) of $G$ is a Nash equilibrium in the mixed extension of $G$. 

**Remark:** The expected utility functions $U_i$ are linear in all mixed strategies.
Support

Intuition:
- It does not make sense to assign positive probability to a strategy that is not a best response to what the other players do.
- Claim: A profile of mixed strategies \( \alpha \) is a Nash equilibrium if and only if everyone only plays best responses to what the others play.

Definition (Support)
Let \( \alpha_i \) be a mixed strategy.
The support of \( \alpha_i \) is the set
\[
\text{supp}(\alpha_i) = \{ a_i \in A_i \mid \alpha_i(a_i) > 0 \}
\]
of actions played with nonzero probability.

Support Lemma

Example (Support lemma)
Matching pennies, strategy profile \( \alpha = (\alpha_1, \alpha_2) \) with
\[
\alpha_1(H) = \frac{2}{3}, \quad \alpha_1(T) = \frac{1}{3}, \quad \alpha_2(H) = \frac{1}{3}, \quad \text{and} \quad \alpha_2(T) = \frac{2}{3}.
\]

For \( \alpha \) to be a Nash equilibrium, both actions in \( \text{supp}(\alpha_2) = \{H, T\} \) have to be best responses to \( \alpha_1 \). Are they?
\[
U_2(\alpha_1, H) = \alpha_1(H) \cdot u_2(H, H) + \alpha_1(T) \cdot u_2(T, H) = \frac{2}{3} \cdot (-1) + \frac{1}{3} \cdot (+1) = -\frac{1}{3},
\]
\[
U_2(\alpha_1, T) = \alpha_1(H) \cdot u_2(H, T) + \alpha_1(T) \cdot u_2(T, T) = \frac{2}{3} \cdot (+1) + \frac{1}{3} \cdot (-1) = \frac{1}{3}.
\]

\( \Rightarrow \) \( H \in \text{supp}(\alpha_2) \), but \( H \notin B_2(\alpha_1) \).
\( \Rightarrow \) \( \alpha \) can not be a Nash equilibrium.

Proof.

\( \Rightarrow \). Let \( \alpha^* \) be a Nash equilibrium with \( a_i \in \text{supp}(\alpha_i^*) \).
Assume that \( a_i \) is not a best response to \( \alpha_{-i}^* \). Because \( U_i \) is linear, player \( i \) can improve his utility by shifting probability in \( \alpha_i^* \) from \( a_i \) to a better response.
This makes the modified \( \alpha_i^* \) a better response than the original \( \alpha_i^* \), i.e., the original \( \alpha_i^* \) was not a best response, which contradicts the assumption that \( \alpha^* \) is a Nash equilibrium.

Support Lemma
Proof (ctd.)

"⇐": Assume that $\alpha^*$ is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy $\alpha'_i$ such that

$$U_i(\alpha^*_{-i}, \alpha'_i) > U_i(\alpha^*_{-i}, \alpha^*_i).$$

Because $U_i$ is linear, there must be a pure strategy

$$\alpha'_i \in \text{supp}(\alpha'_i)$$

that has higher utility than some pure strategy $\alpha^*_i \in \text{supp}(\alpha^*_i)$.

Therefore, $\text{supp}(\alpha'_i)$ does not only contain best responses to $\alpha^*_{-i}$. $\square$

Example (Mixed-strategy Nash equilibria in BoS (ctd.))

**Consequence:** Only need to search for additional Nash equilibria with support sets $\{B, S\}$ vs. $\{B, S\}$.

Assume that $(\alpha^*_1, \alpha^*_2)$ is a Nash equilibrium with $0 < \alpha^*_1(B) < 1$ and $0 < \alpha^*_2(B) < 1$. Then

$$U_1(B, \alpha^*_2) = U_1(S, \alpha^*_2)$$
$$\Rightarrow 2 \cdot \alpha^*_2(B) + 0 \cdot \alpha^*_2(S) = 0 \cdot \alpha^*_2(B) + 1 \cdot \alpha^*_2(S)$$
$$\Rightarrow 2 \cdot \alpha^*_2(B) = 1 - \alpha^*_2(B)$$
$$\Rightarrow 3 \cdot \alpha^*_2(B) = 1$$
$$\Rightarrow \alpha^*_2(B) = \frac{1}{3} \quad \text{(and } \alpha^*_2(S) = \frac{2}{3}).$$

Similarly, we get $\alpha^*_1(B) = \frac{2}{3}$ and $\alpha^*_1(S) = \frac{1}{3}$.

The payoff profile of this equilibrium is $(\frac{2}{3}, \frac{2}{3})$.

Computing Mixed-Strategy Nash Equilibria

Example (Mixed-strategy Nash equilibria in BoS)

<table>
<thead>
<tr>
<th></th>
<th>$B$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>2.1</td>
<td>0.0</td>
</tr>
<tr>
<td>$S$</td>
<td>0.0</td>
<td>1.2</td>
</tr>
</tbody>
</table>

We already know: $(B, B)$ and $(S, S)$ are pure Nash equilibria.

Possible supports (excluding “pure-vs-pure” strategies) are:

- $\{B\}$ vs. $\{B, S\}$,
- $\{S\}$ vs. $\{B, S\}$,
- $\{B, S\}$ vs. $\{B\}$,
- $\{B, S\}$ vs. $\{S\}$

and $\{B, S\}$ vs. $\{B, S\}$

Observation: In Bach or Stravinsky, pure strategies have unique best responses. Therefore, there can be no Nash equilibrium of “pure-vs-strictly-mixed” type.

Support Lemma

**Remark**

Let $G = \langle \{1, 2\}, (A_i), (u_i) \rangle$ with $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$ be a two-player game with two actions each, and $(T, \alpha^*_2)$ with $0 < \alpha^*_2(L) < 1$ be a Nash equilibrium of $G$.

Then at least one of the profiles $(T, L)$ and $(T, R)$ is also a Nash equilibrium of $G$.

**Reason:** Both $L$ and $R$ are best responses to $T$. Assume that $T$ was neither a best response to $L$ nor to $R$. Then $B$ would be a better response than $T$ both to $L$ and to $R$.

With the linearity of $U_1$, $B$ would also be a better response to $\alpha^*_2$ than $T$ is. Contradiction.
Support Lemma

Example
Consider the Nash equilibrium \( \alpha^* = (\alpha^*_1, \alpha^*_2) \) with
\[
\alpha^*_1(T) = 1, \quad \alpha^*_1(B) = 0, \quad \alpha^*_2(L) = 1/10, \quad \alpha^*_2(R) = 9/10
\]
in the following game:

\[
\begin{array}{c|cc}
 & L & R \\
\hline
T & 1, 1 & 1, 1 \\
B & 2, 2 & -5, -5
\end{array}
\]

Here, \((T, R)\) is also a Nash equilibrium.

Nash’s Theorem

Motivation: When does a strategic game have a mixed-strategy Nash equilibrium?

In the previous chapter, we discussed necessary and sufficient conditions for the existence of Nash equilibria for the special case of zero-sum games. Can we make other claims?

Theorem (Nash’s theorem)

Every finite strategic game has a mixed-strategy Nash equilibrium.

Proof sketch.
Consider the set-valued function of best responses
\[
B : \mathbb{R}^{|A_i|} \to 2^{\mathbb{R}^{|A_i|}}
\]
with
\[
B(\alpha) = \prod_{i \in N} B_i(\alpha_i).
\]

A mixed strategy profile \( \alpha \) is a fixed point of \( B \) if and only if \( \alpha \in B(\alpha) \) if and only if \( \alpha \) is a mixed-strategy Nash equilibrium.

The graph of \( B \) has to be connected. Then there is at least one point on the fixpoint diagonal.
Nash’s Theorem

Outline for the formal proof:

1. Review of necessary mathematical definitions
   ~ Subsection “Definitions”
2. Statement of a fixpoint theorem used to prove Nash’s theorem (without proof)
   ~ Subsection “Kakutani’s Fixpoint Theorem”
3. Proof of Nash’s theorem using fixpoint theorem
   ~ Subsection “Proof of Nash’s Theorem”

Definition

A set $X \subseteq \mathbb{R}^n$ is closed if $X$ contains all its limit points, i.e., if $(x_k)_{k \in \mathbb{N}}$ is a sequence of elements in $X$ and $\lim_{k \to \infty} x_k = x$, then also $x \in X$.

Example

Closed: $x \in X$
Not closed: $x \notin X$

Definition

A set $X \subseteq \mathbb{R}^n$ is bounded if for each $i = 1, \ldots, n$ there are lower and upper bounds $a_i, b_i \in \mathbb{R}$ such that

$$X \subseteq \prod_{i=1}^{n} [a_i, b_i].$$

Example

Bounded: $\quad$ Not bounded:

Definition

A set $X \subseteq \mathbb{R}^n$ is convex if for all $x, y \in X$ and all $\lambda \in [0, 1]$,

$$\lambda x + (1 - \lambda) y \in X.$$

Example

Convex: $\quad$ Not convex:
Mixed Strategies
Nash's Theorem
Definitions
Kakutani's Fixpoint Theorem
Proof of Nash's Theorem
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Summary

Nash’s Theorem

Definition
For a function \( f : X \rightarrow 2^X \), the graph of \( f \) is the set
\[
\text{Graph}(f) = \{(x, y) \mid x \in X, y \in f(x)\}.
\]

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Nash’s Theorem

Kakutani’s Fixpoint Theorem

Theorem (Kakutani’s fixpoint theorem)
Let \( X \subseteq \mathbb{R}^n \) be a nonempty, closed, bounded and convex set
and let \( f : X \rightarrow 2^X \) be a function such that
- for all \( x \in X \), the set \( f(x) \subseteq X \) is nonempty and convex, and
- \( \text{Graph}(f) \) is closed.

Then there is an \( x \in X \) with \( x \in f(x) \), i.e., \( f \) has a fixpoint.

Proof.
See Shizuo Kakutani, A generalization of Brouwer’s fixed point theorem, 1941, or your favorite advanced calculus textbook, or the Internet.

For German speakers: Harro Heuser, Lehrbuch der Analysis, Teil 2, also has a proof (Abschnitt 232).

Example
Let \( X = [0, 1] \).

Kakutani’s theorem applicable:

\[
\begin{align*}
&f(x) = \begin{cases} 1 & x < 0.5 \\ 0 & x \geq 0.5 \end{cases} \\
&f(x) = \begin{cases} 0 & x < 0.5 \\ 1 & x \geq 0.5 \end{cases}
\end{align*}
\]

Kakutani’s theorem not applicable:

\[
\begin{align*}
&f(x) = 1 - x \\
&f(x) = x
\end{align*}
\]

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Nash’s Theorem

Proof

Apply Kakutani’s fixpoint theorem using \( X = \mathcal{A} = \prod_{i \in N} \Delta(A_i) \)
and \( f = B \), where \( B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i}) \).

We have to show:
1. \( \mathcal{A} \) is nonempty,
2. \( \mathcal{A} \) is closed,
3. \( \mathcal{A} \) is bounded,
4. \( \mathcal{A} \) is convex,
5. \( B(\alpha) \) is nonempty for all \( \alpha \in \mathcal{A} \),
6. \( B(\alpha) \) is convex for all \( \alpha \in \mathcal{A} \), and
7. \( \text{Graph}(B) \) is closed.

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**Nash’s Theorem**

**Proof**

**Proof (ctd.)**

Some notation:

- Assume without loss of generality that \( N = \{1, \ldots, n\} \).
- A profile of mixed strategies can be written as a vector of \( M = \sum_{i \in N} |A_i| \) real numbers in the interval [0, 1] such that numbers for the same player add up to 1.

For example, \( \alpha = (\alpha_1, \alpha_2) \) with \( \alpha_1(T) = 0.7, \alpha_1(M) = 0.0, \alpha_1(B) = 0.3, \alpha_2(L) = 0.4, \alpha_2(R) = 0.6 \) can be seen as the vector

\[
\begin{pmatrix}
0.7, & 0.0, & 0.3, & 0.4, & 0.6
\end{pmatrix}
\]

This allows us to interpret the set \( \mathscr{A} \) of mixed strategy profiles as a subset of \( \mathbb{R}^M \).

---

**Nash’s Theorem**

**Proof**

**Proof (ctd.)**

1. **\( \mathscr{A} \) nonempty:** Trivial. \( \mathscr{A} \) contains the tuple \((1, 0, \ldots, 0, 1, 0, \ldots, 0)\). | \( |A_1| \) times | \( |A_n| \) times

2. **\( \mathscr{A} \) closed:** Let \( \alpha_1, \alpha_2, \ldots \) be a sequence in \( \mathscr{A} \) that converges to \( \lim_{k \to \infty} \alpha_k = \alpha \). Suppose \( \alpha \notin \mathscr{A} \). Then either there is some component of \( \alpha \) that is less than zero or greater than one, or the components for some player \( i \) add up to a value other than one.

Since \( \alpha \) is a limit point, the same must hold for some \( \alpha_k \) in the sequence. But then, \( \alpha_k \notin \mathscr{A} \), a contradiction. Hence \( \mathscr{A} \) is closed.

---

**Nash’s Theorem**

**Proof**

**Proof (ctd.)**

3. **\( \mathscr{A} \) bounded:** Trivial. All entries are between 0 and 1, i.e., \( \mathscr{A} \) is bounded by \([0, 1]^M\).

4. **\( \mathscr{A} \) convex:** Let \( \alpha, \beta \in \mathscr{A} \) and \( \lambda \in [0, 1] \), and consider \( \gamma = \lambda \alpha + (1 - \lambda)\beta \). Then

\[
\min(\gamma) = \min(\lambda \alpha + (1 - \lambda)\beta) \\
\geq \lambda \cdot \min(\alpha) + (1 - \lambda) \cdot \min(\beta) \\
\geq \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0,
\]

and similarly, \( \max(\gamma) \leq 1 \).

Hence, all entries in \( \gamma \) are still in \([0, 1]\).

---

**Nash’s Theorem**

**Proof**

**Proof (ctd.)**

5. **\( \mathscr{A} \) convex (ctd.):** Let \( \bar{\alpha}, \bar{\beta} \) and \( \bar{\gamma} \) be the sections of \( \alpha, \beta \) and \( \gamma \), respectively, that determine the probability distribution for player \( i \). Then

\[
\sum \bar{\gamma} = \sum (\lambda \bar{\alpha} + (1 - \lambda)\bar{\beta}) \\
= \lambda \cdot \sum \bar{\alpha} + (1 - \lambda) \cdot \sum \bar{\beta} \\
= \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1.
\]

Hence, all probabilities for player \( i \) in \( \gamma \) still sum up to 1. Altogether, \( \gamma \in \mathscr{A} \), and therefore, \( \mathscr{A} \) is convex.
Proof (ctd.)

**B(α) nonempty:** For a fixed $\pi_{-i}$, $U_i$ is linear in the mixed strategies of player $i$, i.e., for $\beta_i, \gamma \in \Delta(A_i)$,

$$U_i(\pi_{-i}, \beta_i + (1 - \lambda) \gamma) = \lambda U_i(\alpha_i, \beta_i) + (1 - \lambda) U_i(\alpha_i, \gamma)$$  

(1)

for all $\lambda \in [0, 1]$.

Hence, $U_i$ is continuous on $\Delta(A_i)$.

Continuous functions on closed and bounded sets take their maximum in that set.

Therefore, $B_i(\pi_{-i}) \neq \emptyset$ for all $i \in N$, and thus $B(\alpha) \neq \emptyset$.

---

**Graph(B) closed (ctd.):** It holds for all $i \in N$:

(D): def. $\alpha_i, \beta_i$; (C) continuity; (B) $\beta_i^k$ best response to $\alpha_i^k$.

---

Proof (ctd.)

**B(α) convex:** This follows, since each $B_i(\pi_{-i})$ is convex.

To see this, let $\alpha_i^k, \alpha_i^{k'} \in B_i(\pi_{-i})$.

Then $U_i(\pi_{-i}, \alpha_i^k) \leq U_i(\pi_{-i}, \alpha_i^{k'})$.

With Equation (1), this implies

$$\lambda \alpha_i^k + (1 - \lambda) \alpha_i^{k'} \in B_i(\pi_{-i}).$$

Hence, $B_i(\pi_{-i})$ is convex.

**Graph(B) closed:** Let $(\alpha^k, \beta^k)$ be a convergent sequence in $Graph(B)$ with $\lim_{k \to \infty} (\alpha^k, \beta^k) = (\alpha, \beta)$.

So, $\alpha^k, \beta^k, \alpha, \beta \in \prod_{i \in N} \Delta(A_i)$ and $\beta^k \in B(\alpha^k)$.

We need to show that $(\alpha, \beta) \in Graph(B)$, i.e., that $\beta \in B(\alpha)$.
Mixed Strategies
Nash’s Theorem
Correlated Equilibria
Summary

3 Correlated Equilibria

Recall: There are three Nash equilibria in Bach or Stravinsky
- \((B, B)\) with payoff profile \((2, 1)\)
- \((S, S)\) with payoff profile \((1, 2)\)
- \((\alpha_1^*, \alpha_2^*)\) with payoff profile \((2/3, 2/3)\) where
  - \(\alpha_1^*(B) = 2/3, \alpha_1^*(S) = 1/3,\)
  - \(\alpha_2^*(B) = 1/3, \alpha_2^*(S) = 2/3.\)

Idea: Use a publicly visible coin toss to decide which action from a mixed strategy is played. This can lead to higher payoffs.

Example (Correlated equilibrium in BoS)

With a fair coin that both players can observe, the players can agree to play as follows:
- If the coin shows heads, both play \(B.\)
- If the coin shows tails, both play \(S.\)

This is stable in the sense that no player has an incentive to deviate from this agreed-upon rule, as long as the other player keeps playing his/her strategy (cf. definition of Nash equilibria).

Expected payoffs: \((3/2, 3/2)\) instead of \((2/3, 2/3)\).

Correlated Equilibria

4 Summary

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Mixed strategies allow randomization.

Characterization of mixed-strategy Nash equilibria:
players only play best responses with positive probability
(support lemma).

Nash's Theorem: Every finite strategic game has a
mixed-strategy Nash equilibrium.

Correlated equilibria can lead to higher payoffs.