Motivation

- Preference relations $\prec$ contain no information about “by how much” one candidate is preferred.
- **Idea**: Use *money* to measure this.
- Use money also for transfers between players “for compensation”.
Setting

Formalization:

- Set of alternatives $A$.
- Set of $n$ players $I$.
- Valuation functions $v_i : A \rightarrow \mathbb{R}$ such that $v_i(a)$ denotes the value player $i$ assigns to alternative $a$.
- Payment functions specifying amount $p_i \in \mathbb{R}$ that player $i$ pays.
- Utility of player $i$: $u_i(a) = v_i(a) - p_i$. 

Second Price Auctions
Incentive Compatible Mechanisms
VCG Mechanisms
Second Price Auctions
Second price auctions:

- There are $n$ players bidding for a single item.
- Player $i$’s private valuations of item: $w_i$.
- Desired outcome: Player with highest private valuation wins bid.
- Players should reveal their valuations truthfully.
- Winner $i$ pays price $p^*$ and has utility $w_i - p^*$.
- Non-winners pay nothing and have utility 0.
Second Price Auctions

Formally:

- \( A = N \)
- \( v_i(a) = \begin{cases} w_i & \text{if } a = i \\ 0 & \text{else} \end{cases} \)

What about payments? Say player \( i \) wins:

- \( p^* = 0 \) (winner pays nothing): bad idea, players would manipulate and publicly declare values \( w'_i \gg w_i \).
- \( p^* = w_i \) (winner pays his valuation): bad idea, players would manipulate and publicly declare values \( w'_i = w_i - \varepsilon \).
- better: \( p^* = \max_{j \neq i} w_j \) (winner pays second highest bid).
Vickrey Auction

Definition (Vickrey Auction)

The winner of the Vickrey Auction (aka second price auction) is the player \( i \) with the highest declared value \( w_i \). He has to pay the second highest declared bid \( p^* = \max_{j \neq i} w_j \).

Proposition (Vickrey)

Let \( i \) be one of the players and \( w_i \) his valuation for the item, \( u_i \) his utility if he truthfully declares \( w_i \) as his valuation of the item, and \( u'_i \) his utility if he falsely declares \( w'_i \) as his valuation of the item. Then \( u_i \geq u'_i \).

Proof

See

Definition (Vickrey Auction)

The winner of the Vickrey Auction (aka second price auction) is the player $i$ with the highest declared value $w_i$. He has to pay the second highest declared bid $p^* = \max_{j \neq i} w_j$.

Proposition (Vickrey)

Let $i$ be one of the players and $w_i$ his valuation for the item, $u_i$ his utility if he truthfully declares $w_i$ as his valuation of the item, and $u'_i$ his utility if he falsely declares $w'_i$ as his valuation of the item. Then $u_i \geq u'_i$.

Proof

Incentive Compatible Mechanisms
Idea: Generalization of Vickrey auctions.

Preferences modeled as functions $v_i : A \to \mathbb{R}$.

Let $V_i$ be the space of all such functions for player $i$.

Unlike for social choice functions: Not only decide about chosen alternative, but also about payments.
**Mechanisms**

---

**Definition (Mechanism)**

A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) consists of

- a social choice function \( f : V_1 \times \cdots \times V_n \rightarrow A \) and
- for each player \( i \), a payment function \( p_i : V_1 \times \cdots \times V_n \rightarrow \mathbb{R} \).

---

**Definition (Incentive Compatibility)**

A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) is called incentive compatible if for each player \( i = 1, \ldots, n \), for all preferences \( v_1 \in V_1, \ldots, v_n \in V_n \) and for each preference \( v'_i \in V_i \),

\[
v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}).
\]
Mechanisms

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A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) consists of

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\]
VCG Mechanisms
VCG Mechanisms

- If \( \langle f, p_1, \ldots, p_n \rangle \) is incentive compatible, truthfully declaring ones preference is dominant strategy.

- The Vickrey-Clarke-Groves mechanism is an incentive compatible mechanism that maximizes “social welfare”, i.e., the sum over all individual utilities \( \sum_{i=1}^{n} v_i(a) \).

- Idea: Reflect other players’ utilities in payment functions, align all players’ incentives with goal of maximizing social welfare.
VCG Mechanisms

Definition (Vickrey-Clarke-Groves mechanism)

A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) is called a Vickrey-Clarke-Groves mechanism (VCG mechanism) if

1. \( f(v_1, \ldots, v_n) \in \arg\max_{a \in A} \sum_{i=1}^{n} v_i(a) \) for all \( v_1, \ldots, v_n \) and
2. there are functions \( h_1, \ldots, h_n \) with \( h_i : V_{-i} \rightarrow \mathbb{R} \) such that \( p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \ldots, v_n)) \) for all \( i = 1, \ldots, n \) and \( v_1, \ldots, v_n \).

Note: \( h_i(v_{-i}) \) independent of player \( i \)'s declared preference \( \Rightarrow \)
\( h_i(v_{-i}) = c \) constant from player \( i \)'s perspective.

Utility of player \( i \) = \( v_i(f(v_1, \ldots, v_n)) + \sum_{j \neq i} v_j(f(v_1, \ldots, v_n)) - c = \sum_{j=1}^{n} v_j(f(v_1, \ldots, v_n)) - c = \) social welfare \(- c.\)
Definition (Vickrey-Clarke-Groves mechanism)

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   \[
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Utility of player \( i \)
\[
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\]
Theorem (Vickrey-Clarke-Groves)

Every VCG mechanism is incentive compatible.

Proof

Let \( i, v_i, v'_i \) and \( v'_i \) be given. Show: Declaring true preference \( v_i \) dominates declaring false preference \( v'_i \).

Let \( a = f(v_i, v_{-i}) \) and \( a' = f(v'_i, v_{-i}) \).

Utility player \( i = \begin{cases} v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i}) & \text{if declaring } v_i \\ v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) & \text{if declaring } v'_i \end{cases} \)

Alternative \( a = f(v_i, v_{-i}) \) maximizes social welfare

\[ v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a'). \]

\[ v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}). \]
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\[ \Rightarrow v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a'). \]

\[ \Rightarrow v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}). \]

\[ \square \]
Theorem (Vickrey-Clarke-Groves)

Every VCG mechanism is incentive compatible.

Proof

Let $i$, $v_{-i}$, $v_i$ and $v'_i$ be given. Show: Declaring true preference $v_i$ dominates declaring false preference $v'_i$.

Let $a = f(v_i, v_{-i})$ and $a' = f(v'_i, v_{-i})$.

Utility player $i = \begin{cases} v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i}) & \text{if declaring } v_i \\ v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) & \text{if declaring } v'_i \end{cases}$

Alternative $a = f(v_i, v_{-i})$ maximizes social welfare

$\Rightarrow v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a')$.

$\Rightarrow v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i})$. 

VCG Mechanisms

Theorem (Vickrey-Clarke-Groves)
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Let \( i, v_{-i}, v_i \) and \( v'_i \) be given. Show: Declaring true preference \( v_i \) dominates declaring false preference \( v'_i \).

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Alternative \( a = f(v_i, v_{-i}) \) maximizes social welfare
\[ v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a'). \]

\[ v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}). \]
Clarke Pivot Rule

- **So far:** payment functions $p_i$ and functions $h_i$ unspecified.
- **One possibility:** $h_i(v_{−i}) = 0$ for all $h_i$ and $v_{−i}$.
  
  **Drawback:** Too much money distributed among players (more than necessary).
- **Further requirements:**
  - Players should pay at most as much as they value the outcome.
  - Players should only pay, never receive money.
Definition (individual rationality)

A mechanism is individually rational if all players always get a nonnegative utility, i.e., if for all \( i = 1, \ldots, n \) and all \( v_1, \ldots, v_n \),

\[
v_i(f(v_1, \ldots, v_n)) - p_i(v_1, \ldots, v_n) \geq 0.
\]

Definition (positive transfers)

A mechanism has no positive transfers if no player is ever paid money, i.e., for all preferences \( v_1, \ldots, v_n \),

\[
p_i(v_1, \ldots, v_n) \geq 0.
\]
Clarke Pivot Function

**Definition (Clarke pivot function)**

The Clarke pivot function is the function

\[ h_i(v_{-i}) = \max_{b \in A} \sum_{j \neq i} v_j(b). \]

This leads to payment functions

\[ p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \]

for \( a = f(v_1, \ldots, v_n) \).

Player \( i \) pays the difference between what the other players could achieve without him and what they achieve with him.

Each player internalizes the externalities he causes.
Clarke Pivot Function

Example

- Players $I = \{1, 2\}$, alternatives $A = \{a, b\}$.
- Values: $v_1(a) = 10$, $v_1(b) = 2$, $v_2(a) = 9$ and $v_2(b) = 15$.
- Without player 1: $b$ best, since $v_2(b) = 15 > 9 = v_2(a)$.
- With player 1: $a$ best, since $v_1(a) + v_2(a) = 10 + 9 = 19 > 10 + 9 = v_1(b) + v_2(b)$.
- With player 1, other players (i.e., player 2) lose $v_2(b) - v_2(a) = 6$ units of utility.

$\Rightarrow$ Clarke pivot function $h_1(v_2) = 15$

$\Rightarrow$ payment function

$$p_1(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq 1} v_j(b) - \sum_{j \neq 1} v_j(a) = 15 - 9 = 6.$$
Players $I = \{1, 2\}$, alternatives $A = \{a, b\}$.

Values: $v_1(a) = 10$, $v_1(b) = 2$, $v_2(a) = 9$ and $v_2(b) = 15$.

Without player 1: $b$ best, since $v_2(b) = 15 > 9 = v_2(a)$.

With player 1: $a$ best, since

$v_1(a) + v_2(a) = 10 + 9 = 19 > 17 = 2 + 15 = v_1(b) + v_2(b)$.

With player 1, other players (i.e., player 2) lose

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$\Rightarrow$ **Clarke pivot function** $h_1(v_2) = 15$

$\Rightarrow$ **payment function**

$$p_1(v_1, \ldots, v_n) = \max_{b \in A} \left( \sum_{j \neq 1} v_j(b) - \sum_{j \neq 1} v_j(a) \right) = 15 - 9 = 6.$$
Clarke Pivot Function

Example

- Players \( I = \{1, 2\} \), alternatives \( A = \{a, b\} \).
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  v_2(b) - v_2(a) = 6 \text{ units of utility}.
  \]
  \[\Rightarrow\] Clarke pivot function \( h_1(v_2) = 15 \)
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⇒ Clarke pivot function \( h_1(v_2) = 15 \)
⇒ payment function

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  \[ v_2(b) - v_2(a) = 6 \text{ units of utility}. \]

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$\Rightarrow$ Clarke pivot function $h_1(v_2) = 15$

$\Rightarrow$ payment function

$$p_1(v_1,\ldots,v_n) = \max_{b \in A} \sum_{j \neq 1} v_j(b) - \sum_{j \neq 1} v_j(a) = 15 - 9 = 6.$$
A VCG mechanism with Clarke pivot functions has no positive transfers. If $v_i(a) \geq 0$ for all $i = 1, \ldots, n$, $v_i \in V_i$ and $a \in A$, then the mechanism is also individually rational.

**Proof**

Let $a = f(v_1, \ldots, v_n)$ be the alternative maximizing $\sum_{j=1}^{n} v_j(a)$, and $b$ the alternative maximizing $\sum_{j \neq i} v_j(b)$.

Utility of player $i$: $u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b)$.

Payment function for $i$: $p_i(v_1, \ldots, v_n) = \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$.

Since $b$ maximizes $\sum_{j \neq i} v_j(b)$: $p_i(v_1, \ldots, v_n) \geq 0$  
(no positive transfers).
Lemma (Clarke pivot rule)

A VCG mechanism with Clarke pivot functions has no positive transfers. If \( v_i(a) \geq 0 \) for all \( i = 1, \ldots, n \), \( v_i \in V_i \) and \( a \in A \), then the mechanism is also individually rational.

Proof

Let \( a = f(v_1, \ldots, v_n) \) be the alternative maximizing \( \sum_{j=1}^{n} v_j(a) \), and \( b \) the alternative maximizing \( \sum_{j \neq i} v_j(b) \).

Utility of player \( i \): \( u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \).

Payment function for \( i \): \( p_i(v_1, \ldots, v_n) = \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \).

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Let \( a = f(v_1, \ldots, v_n) \) be the alternative maximizing \( \sum_{j=1}^{n} v_j(a) \), and \( b \) the alternative maximizing \( \sum_{j \neq i} v_j(b) \).

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A VCG mechanism with Clarke pivot functions has no positive transfers. If \( v_i(a) \geq 0 \) for all \( i = 1, \ldots, n, \) \( v_i \in V_i \) and \( a \in A \), then the mechanism is also individually rational.

Proof

Let \( a = f(v_1, \ldots, v_n) \) be the alternative maximizing \( \sum_{j=1}^{n} v_j(a) \), and \( b \) the alternative maximizing \( \sum_{j \neq i} v_j(b) \).

Utility of player \( i \):
\[
u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b).
\]

Payment function for \( i \):
\[
p_i(v_1, \ldots, v_n) = \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a).
\]

Since \( b \) maximizes \( \sum_{j \neq i} v_j(b) \):
\[
p_i(v_1, \ldots, v_n) \geq 0
\]
(no positive transfers).
Individual rationality: Since \( v_i(b) \geq 0 \),

\[
    u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_{j=1}^{n} v_j(a) - \sum_{j=1}^{n} v_j(b).
\]

Since \( a \) maximizes \( \sum_{j=1}^{n} v_j(a) \),

\[
    \sum_{j=1}^{n} v_j(a) \geq \sum_{j=1}^{n} v_j(b)
\]

and hence \( u_i \geq 0 \).

Therefore, the mechanism is also individually rational.
Clarke Pivot Rule

Proof (ctd.)

Individual rationality: Since $v_i(b) \geq 0$,

$$u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_{j=1}^{n} v_j(a) - \sum_{j=1}^{n} v_j(b).$$

Since $a$ maximizes $\sum_{j=1}^{n} v_j(a)$,

$$\sum_{j=1}^{n} v_j(a) \geq \sum_{j=1}^{n} v_j(b)$$

and hence $u_i \geq 0$.

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Proof (ctd.)

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$$u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_{j=1}^n v_j(a) - \sum_{j=1}^n v_j(b).$$

Since $a$ maximizes $\sum_{j=1}^n v_j(a)$,

$$\sum_{j=1}^n v_j(a) \geq \sum_{j=1}^n v_j(b)$$

and hence $u_i \geq 0$.

Therefore, the mechanism is also individually rational.
Vickrey Auction as a VCG Mechanism

- $A = N$. Valuations: $w_i, v_a(a) = w_a, v_i(a) = 0$ ($i \neq a$).

- $a$ maximizes social welfare $\sum_{i=1}^{n} v_i(a)$ iff $a$ maximizes $w_a$.

- Let $a = f(v_1, \ldots, v_n) = \arg\max_{j \in A} w_j$ be the highest bidder.

- Payments: $p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$.

- But $\max_{b \in A} \sum_{j \neq i} v_j(b) = \max_{b \in A \setminus \{i\}} w_b$.

- Winner pays value of second highest bid:

$$p_a(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq a} v_j(b) - \sum_{j \neq a} v_j(a)$$

$$= \max_{b \in A \setminus \{a\}} w_b - 0 = \max_{b \in A \setminus \{a\}} w_b.$$ 

- Non-winners pay nothing: For $i \neq a$,

$$p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$$

$$= \max_{b \in A \setminus \{i\}} w_b - w_a = w_a - w_a = 0.$$
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- Winner pays value of second highest bid:

  $$p_a(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq a} v_j(b) - \sum_{j \neq a} v_j(a)$$

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Vickrey Auction as a VCG Mechanism

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- **Winner pays value of second highest bid:**
  \[
p_a(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq a} v_j(b) - \sum_{j \neq a} v_j(a)
  = \max_{b \in A \setminus \{a\}} w_b - 0 = \max_{b \in A \setminus \{a\}} w_b.
  \]
- **Non-winners pay nothing:** For \( i \neq a \),
  \[
p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)
  = \max_{b \in A \setminus \{i\}} w_b - w_a = w_a - w_a = 0.
  \]
Example: Bilateral Trade

- **Seller** $s$ offers item he values with $0 \leq w_s \leq 1$.
- Potential **buyer** $b$ values item with $0 \leq w_b \leq 1$.
- Alternatives $A = \{\text{trade}, \text{no-trade}\}$.
- Valuations:
  
  $v_s(\text{no-trade}) = 0, \quad v_s(\text{trade}) = -w_s,$
  
  $v_b(\text{no-trade}) = 0, \quad v_b(\text{trade}) = w_b.$

- VCG mechanism maximizes $v_s(a) + v_b(a)$.
- We have
  
  $v_s(\text{trade}) + v_b(\text{trade}) = w_b - w_s,$
  
  $v_s(\text{no-trade}) + v_b(\text{no-trade}) = 0$

  i.e., **trade** maximizes social welfare iff $w_b \geq w_s$. 
Example: Bilateral Trade (ctd.)

■ **Requirement:** if *no-trade* is chosen, neither player pays anything:

\[ p_s(v_s, v_b) = p_b(v_s, v_b) = 0. \]

■ To that end, choose Clarke pivot function for **buyer**:

\[ h_b(v_s) = \max_{a \in A} v_s(a). \]

■ **For seller:** Modify Clarke pivot function by an additive constant and set

\[ h_s(v_b) = \max_{a \in A} v_b(a) - w_b. \]
Example: Bilateral Trade (ctd.)

- For alternative *no-trade*,

\[
p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\text{no-trade})
\]

\[
= w_b - w_b - 0 = 0 \quad \text{and}
\]

\[
p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\text{no-trade})
\]

\[
= 0 - 0 = 0.
\]

- For alternative *trade*,

\[
p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\text{trade})
\]

\[
= w_b - w_b - w_b = -w_b \quad \text{and}
\]

\[
p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\text{trade})
\]

\[
= 0 + w_s = w_s.
\]
Because $w_b \geq w_s$, the seller gets at least as much as the buyer pays, i.e., the mechanism subsidizes the trade.

Without subsidies, no incentive compatible bilateral trade possible.

Note: Buyer and seller can exploit the system by colluding.
Example: Public Project

- Project costs $C$ units.
- Each citizen $i$ privately values the project at $w_i$ units.
- Government will undertake project if $\sum_i w_i > C$.
- Alternatives: $A = \{\text{project, no-project}\}$.
- Valuations:
  \[
  v_G(\text{project}) = -C, \quad v_G(\text{no-project}) = 0, \\
  v_i(\text{project}) = w_i, \quad v_i(\text{no-project}) = 0.
  \]

- VCG mechanism with Clarke pivot rule: for each citizen $i$,
  \[
  h_i(v_{-i}) = \max_{a \in A} \left( \sum_{j \neq i} v_j(a) + v_G(a) \right) \\
  = \begin{cases} 
  \sum_{j \neq i} w_j - C, & \text{if } \sum_{j \neq i} w_j > C \\
  0, & \text{otherwise}.
  \end{cases}
  \]
Example: Public Project (ctd.)

- **Citizen** $i$ pivotal if $\sum_j w_j > C$ and $\sum_{j \neq i} w_j \leq C$.
- Payment function for citizen $i$:

  $$p_i(v_{1..n}, v_G) = h_i(v_{-i}) - \left( \sum_{j \neq i} v_j (f(v_{1..n}, v_G)) + v_G (f(v_{1..n}, v_G)) \right)$$

  - **Case 1**: Project undertaken, $i$ pivotal:
    $$p_i(v_{1..n}, v_G) = 0 - \left( \sum_{j \neq i} w_j - C \right) = C - \sum_{j \neq i} w_j$$
  
  - **Case 2**: Project undertaken, $i$ not pivotal:
    $$p_i(v_{1..n}, v_G) = \left( \sum_{j \neq i} w_j - C \right) - \left( \sum_{j \neq i} w_j - C \right) = 0$$
  
  - **Case 3**: Project not undertaken:
    $$p_i(v_{1..n}, v_G) = 0$$
Example: Public Project (ctd.)

- I.e., citizen $i$ pays nonzero amount

\[ C - \sum_{j \neq i} w_j \]

only if he is pivotal.

- He pays difference between value of project to fellow citizens and cost $C$, in general less than $w_i$.

- Generally,

\[ \sum_i p_i(\text{project}) \leq C \]

i.e., project has to be subsidized.
Example: Buying a Path in a Network

- Communication network modeled as $G = (V, E)$.
- Each link $e \in E$ owned by different player $e$.
- Each link $e \in E$ has cost $c_e$ if used.
- **Objective**: procure communication path from $s$ to $t$.
- **Alternatives**: $A = \{ p \mid p \text{ path from } s \text{ to } t \}$.
- **Valuations**: $v_e(p) = -c_e$, if $e \in p$, and $v_e(p) = 0$, if $e \notin p$.
- **Maximizing social welfare**: minimize $\sum_{e \in p} c_e$ over all paths $p$ from $s$ to $t$.
- **Example**:

$\begin{align*}
\text{graph} & : \\
\text{nodes} & : s, i, t \\
\text{edges} & : (s, i, a), (i, t, d), (i, s, b), (t, i, e) \\
\text{costs} & : c_a = 4, c_b = 3, c_d = 12, c_e = 5
\end{align*}$
Example: Buying a Path in a Network (ctd.)

- For $G = (V, E)$ and $e \in E$ let $G \setminus e = (V, E \setminus \{e\})$.
- **VGC mechanism with Clarke pivot function:**

$$h_e(v-e) = \max_{p' \in G \setminus e} \sum_{e' \in p'} -c_{e'}$$

i.e., the cost of the cheapest path from $s$ to $t$ in $G \setminus e$.
(Assume that $G$ is 2-connected, s.t. such $p'$ exists.)

- **Payment functions:** for chosen path $p = f(v_1, \ldots, v_n)$,

$$p_e(v_1, \ldots, v_n) = h_e(v-e) - \sum_{e \notin e' \in p} -c_{e'}.$$  

- **Case 1:** $e \notin p$. Then $p_e(v_1, \ldots, v_n) = 0$.
- **Case 2:** $e \in p$. Then

$$p_e(v_1, \ldots, v_n) = \max_{p' \in G \setminus e} \sum_{e' \in p'} -c_{e'} - \sum_{e \notin e' \in p} -c_{e'}.$$
Example: Buying a Path in a Network (ctd.)

- **Example:**

\[
\begin{array}{ccc}
\text{Cost along } b \text{ and } e: & 8 \\
\text{Cost without } e: & 3 \\
\text{Cost of cheapest path without } e: & 15 \text{ (along } b \text{ and } d) \\
\text{Difference is payment: } & -15 - (-3) = -12 \\
\text{I.e., owner of arc } e \text{ gets payed } 12 \text{ for using his arc.}
\end{array}
\]

- **Note:** Alternative path after deletion of \( e \) does not necessarily differ from original path at only one position. Could be totally different.
Summary

- New preference model: with money.
- VCG mechanisms generalize Vickrey auctions.
- VCG mechanisms are incentive compatible mechanisms maximizing social welfare.
- With Clarke pivot rule: even no positive transfers and individually rational (if nonnegative valuations).
- Various application areas.