Motivation

- Preference relations $\prec$ contain no information about "by how much" one candidate is preferred.
- Idea: Use money to measure this.
- Use money also for transfers between players "for compensation".

Setting

Formalization:
- Set of alternatives $A$.
- Set of $n$ players $I$.
- Valuation functions $v_i : A \rightarrow \mathbb{R}$ such that $v_i(a)$ denotes the value player $i$ assigns to alternative $a$.
- Payment functions specifying amount $p_i \in \mathbb{R}$ that player $i$ pays.
- Utility of player $i$: $u_i(a) = v_i(a) - p_i$.

Second Price Auctions

Second price auctions:
- There are $n$ players bidding for a single item.
- Player $i$'s private valuations of item: $w_i$.
- Desired outcome: Player with highest private valuation wins bid.
- Players should reveal their valuations truthfully.
- Winner $i$ pays price $p^*$ and has utility $w_i - p^*$.
- Non-winners pay nothing and have utility 0.
Second Price Auctions

Formally:
- \( A = N \)
- \( v_i(a) = \begin{cases} w_i & \text{if } a = i \\ 0 & \text{else} \end{cases} \)

What about payments? Say player \( i \) wins:
- \( p^* = 0 \) (winner pays nothing): bad idea, players would manipulate and publicly declare values \( w_i' \gg w_i \).
- \( p^* = w_i \) (winner pays his valuation): bad idea, players would manipulate and publicly declare values \( w_i' = w_i - \varepsilon \).
- better: \( p^* = \max_{j \neq i} w_j \) (winner pays second highest bid).

Vickrey Auction

Definition (Vickrey Auction)
The winner of the Vickrey Auction (aka second price auction) is the player \( i \) with the highest declared value \( w_i \). He has to pay the second highest declared bid \( p^* = \max_{j \neq i} w_j \).

Proposition (Vickrey)
Let \( i \) be one of the players and \( w_i \) his valuation for the item, \( u_i \) his utility if he truthfully declares \( w_i \) as his valuation of the item, and \( u_i' \) his utility if he falsely declares \( w_i' \) as his valuation of the item. Then \( u_i \geq u_i' \).

Proof

Incentive Compatible Mechanisms

Idea: Generalization of Vickrey auctions.
Preferences modeled as functions \( v_i : A \to \mathbb{R} \).
Let \( V_i \) be the space of all such functions for player \( i \).
Unlike for social choice functions: Not only decide about chosen alternative, but also about payments.

Mechanisms

Definition (Mechanism)
A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) consists of:
- a social choice function \( f : V_1 \times \cdots \times V_n \to A \) and
- for each player \( i \), a payment function \( p_i : V_1 \times \cdots \times V_n \to \mathbb{R} \).

Definition (Incentive Compatibility)
A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) is called incentive compatible if for each player \( i = 1, \ldots, n \), for all preferences \( v_1 \in V_1, \ldots, v_n \in V_n \) and for each preference \( v_i' \in V_i \),
\[
v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i}).
\]
VCG Mechanisms

- If \( (f,p_1,\ldots,p_n) \) is incentive compatible, truthfully declaring ones preference is dominant strategy.

- The Vickrey-Clarke-Groves mechanism is an incentive compatible mechanism that maximizes “social welfare”, i.e., the sum over all individual utilities \( \sum_{i=1}^{n} v_i(a) \).

- Idea: Reflect other players’ utilities in payment functions, align all players’ incentives with goal of maximizing social welfare.

VCG Mechanisms

Theorem (Vickrey-Clarke-Groves mechanism)

Every VCG mechanism is incentive compatible.

Proof

Let \( i, v_{-i}, v_i \) and \( v_i' \) be given. Show: Declaring true preference \( v_i \) dominates declaring false preference \( v_i' \).

Let \( a = f(v_i, v_{-i}) \) and \( a' = f(v_i', v_{-i}) \).

Utility player \( i = v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i}) \) if declaring \( v_i \)

\[ \begin{align*}
  v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) & \quad \text{if declaring } v_i' \\
  v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) & \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i}).
\end{align*} \]

VCG Mechanisms

Definition (Vickrey-Clarke-Groves mechanism)

A mechanism \( (f,p_1,\ldots,p_n) \) is called a Vickrey-Clarke-Groves mechanism (VCG mechanism) if

1. \( f(v_1,\ldots,v_n) \in \text{argmax}_{a \in A} \sum_{i=1}^{n} v_i(a) \) for all \( v_1,\ldots,v_n \) and
2. there are functions \( h_1,\ldots,h_n \) with \( h_i : V_{-i} \rightarrow \mathbb{R} \) such that \( p_i(v_1,\ldots,v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1,\ldots,v_n)) \) for all \( i = 1,\ldots,n \) and \( v_1,\ldots,v_n \).

Note: \( h_i(v_{-i}) \) independent of player \( i \)'s declared preference \( \Rightarrow h_i(v_{-i}) = c \) constant from player \( i \)'s perspective.

Utility of player \( i = v_i(f(v_1,\ldots,v_n)) + \sum_{j \neq i} v_j(f(v_1,\ldots,v_n)) - c = \sum_{i=1}^{n} v_i(a) - c = \text{social welfare} - c. \)

VCG Mechanisms

Clarke Pivot Rule

- So far: payment functions \( p_i \) and functions \( h_i \) unspecified.
- One possibility: \( h_i(v_{-i}) = 0 \) for all \( h_i \) and \( v_{-i} \).

  Drawback: Too much money distributed among players (more that necessary).

- Further requirements:
  - Players should pay at most as much as they value the outcome.
  - Players should only pay, never receive money.
Individual Rationality, Positive Transfers

Definition (individual rationality)
A mechanism is individually rational if all players always get a nonnegative utility, i.e., if for all \( i = 1, \ldots, n \) and all \( v_1, \ldots, v_n \),
\[
v_i(f(v_1, \ldots, v_n)) - p_i(v_1, \ldots, v_n) \geq 0.
\]

Definition (positive transfers)
A mechanism has no positive transfers if no player is ever paid money, i.e., for all preferences \( v_1, \ldots, v_n \),
\[
p_i(v_1, \ldots, v_n) \geq 0.
\]

Clarke Pivot Function

Example
- Players \( I = \{1, 2\} \), alternatives \( A = \{a, b\} \).
- Values: \( v_1(a) = 10, v_1(b) = 2, v_2(a) = 9 \) and \( v_2(b) = 15 \).
- Without player 1: \( b \) best, since \( v_2(b) = 15 > 9 = v_2(a) \).
- With player 1: \( a \) best, since \( v_1(a) + v_2(a) = 10 + 9 = 19 > 2 + 15 = v_1(b) + v_2(b) \).
- With player 1, other players (i.e., player 2) lose \( v_2(b) - v_2(a) = 6 \) units of utility.

⇒ Clarke pivot function \( h_1(v_2) = 15 \)
⇒ payment function
\[
p_1(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) = 15 - 9 = 6.
\]

Clarke Pivot Rule

Lemma (Clarke pivot rule)
A VCG mechanism with Clarke pivot functions has no positive transfers. If \( v_i(a) \geq 0 \) for all \( i = 1, \ldots, n \), \( v_i \in V_i \) and \( a \in A \), then the mechanism is also individually rational.

Proof
Let \( a = f(v_1, \ldots, v_n) \) be the alternative maximizing \( \sum_{j=1}^n v_j(a) \), and \( b \) the alternative maximizing \( \sum_{j \neq i} v_j(b) \).

Utility of player \( i \): \( u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \).

Payment function for \( i \): \( p_i(v_1, \ldots, v_n) = \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \).

Since \( b \) maximizes \( \sum_{j \neq i} v_j(b) \): \( p_i(v_1, \ldots, v_n) \geq 0 \)
(no positive transfers).
Therefore, the mechanism is also individually rational.

**Proof (ctd.)**

Individual rationality: Since $v_i(b) \geq 0$,

$$u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_{j = 1}^n v_j(a) - \sum_{j = 1}^n v_j(b).$$

Since $a$ maximizes $\sum_{j = 1}^n v_j(a)$,

$$\sum_{j = 1}^n v_j(a) \geq \sum_{j = 1}^n v_j(b)$$

and hence $u_i \geq 0$.

Therefore, the mechanism is also individually rational.

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**Example: Bilateral Trade**

- **Seller** offers item he values with $0 \leq w_s \leq 1$.
- Potential **buyer** values item with $0 \leq w_b \leq 1$.
- Alternatives $A = \{\text{trade, no-trade}\}$.
- Valuations:

  $$v_s(\text{no-trade}) = 0, \quad v_s(\text{trade}) = -w_s,$$
  $$v_b(\text{no-trade}) = 0, \quad v_b(\text{trade}) = w_b.$$  

- VCG mechanism maximizes $v_s(a) + v_b(a)$.
- We have

  $$v_s(\text{trade}) + v_b(\text{trade}) = w_b - w_s,$$
  $$v_s(\text{no-trade}) + v_b(\text{no-trade}) = 0$$

  i.e., **trade** maximizes social welfare iff $w_b \geq w_s$.

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**Vickrey Auction as a VCG Mechanism**

- $A = N$. Valuations: $v_i(a) = w_a, v_i(a) = 0 (i \neq a)$.
- $a$ maximizes social welfare $\sum_{i \in A} v_i(a)$ if $a$ maximizes $w_a$.
- Let $a = f(v_1, \ldots, v_n) = \arg\max_{j \in A} w_j$ be the highest bidder.
- Payments: $p_i(v_1, \ldots, v_n) = \max_{b \in A \setminus \{j\}} v_j(b) - \sum_{j \neq i} v_j(a)$.
- Winner pays value of second highest bid:

  $$p_a(v_1, \ldots, v_n) = \max_{b \in A \setminus \{a\}} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$$

  $$= \max_{b \in A \setminus \{a\}} w_b - 0 = \max_{b \in A \setminus \{a\}} w_b.$$
- Non-winners pay nothing: For $i \neq a$,

  $$p_i(v_1, \ldots, v_n) = \max_{b \in A \setminus \{i\}} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$$

  $$= \max_{b \in A \setminus \{i\}} w_b - w_a = w_a - w_a = 0.$$
Example: Public Project (ctd.)

- For alternative \textit{no-trade},
  \[
p_s(v_i, v_b) = \max_{a \in A} v_p(a) - w_b - v_b(\text{no-trade})
  = w_b - w_b - 0 = 0 \quad \text{and}
  p_b(v_i, v_b) = \max_{a \in A} v_p(a) - v_i(\text{no-trade})
  = 0 - 0 = 0.
\]

- For alternative \textit{trade},
  \[
p_s(v_i, v_b) = \max_{a \in A} v_p(a) - w_b - v_b(\text{trade})
  = w_b - w_b - w_b = -w_b \quad \text{and}
  p_b(v_i, v_b) = \max_{a \in A} v_p(a) - v_i(\text{trade})
  = 0 + w_s = w_s.
\]

Example: Bilateral Trade (ctd.)

- Because \(w_b \geq w_s\), the seller gets at least as much as the buyer pays, i.e., the mechanism \textit{subsidizes} the trade.
- Without subsidies, no incentive compatible bilateral trade possible.
- \textbf{Note}: Buyer and seller can exploit the system by \textit{colluding}.

Example: Public Project

- Project costs \(C\) units.
- Each citizen \(i\) privately values the project at \(w_i\) units.
- Government will undertake project if \(\sum w_i > C\).
- Alternatives: \(A = \{\text{project, no-project}\}\).
- Valuations:
  \[
  v_G(\text{project}) = -C, \quad v_G(\text{no-project}) = 0,
  v_i(\text{project}) = w_i, \quad v_i(\text{no-project}) = 0.
  \]
- VCG mechanism with Clarke pivot rule: for each citizen \(i\),
  \[
  h_i(v_{-i}) = \max_{a \in A} \left( \sum_{j \neq i} v_j(a) + v_G(a) \right)
  = \begin{cases} 
  \sum_{j \neq i} w_j - C, & \text{if } \sum_{j \neq i} w_j > C \\
  0, & \text{otherwise}.
  \end{cases}
  \]

Example: Bilateral Trade (ctd.)

- Citizen \(i\) \textit{pivotal} if \(\sum w_j > C\) and \(\sum_{j \neq i} w_j \leq C\).
- Payment function for citizen \(i\):
  \[
  p_i(v_{1..n}, v_G) = h_i(v_{-i}) - \left( \sum_{j \neq i} v_j(f(v_{1..n}, v_G)) + v_G(f(v_{1..n}, v_G)) \right)
  \]
  \[
  \text{Case 1: Project undertaken, } i \text{ pivotal:}
  p_i(v_{1..n}, v_G) = 0 - \left( \sum_{j \neq i} w_j - C \right) = C - \sum_{j \neq i} w_j
  \]
  \[
  \text{Case 2: Project undertaken, } i \text{ not pivotal:}
  p_i(v_{1..n}, v_G) = \left( \sum_{j \neq i} w_j - C \right) - \left( \sum_{j \neq i} w_j - C \right) = 0
  \]
  \[
  \text{Case 3: Project not undertaken:}
  p_i(v_{1..n}, v_G) = 0
  \]
Example: Public Project (ctd.)

- i.e., citizen $i$ pays nonzero amount
  \[ C - \sum_{j \neq i} w_j \]
  only if he is pivotal.
- He pays difference between value of project to fellow citizens and cost $C$, in general less than $w_i$.
- Generally,
  \[ \sum_i p_i(\text{project}) \leq C \]
  i.e., project has to be subsidized.

Example: Buying a Path in a Network (ctd.)

- Communication network modeled as $G = (V, E)$.
- Each link $e \in E$ owned by different player $e$.
- Each link $e \in E$ has cost $c_e$, if used.
- Objective: procure communication path from $s$ to $t$.
- Alternatives: $A = \{p \mid p \text{ path from } s \text{ to } t\}$.
- Valuations: $v_e(p) = -c_e$, if $e \in p$, and $v_e(p) = 0$, if $e \notin p$.
- Maximizing social welfare:
  \[ \min \sum_{e \in p} c_e \text{ over all paths } p \text{ from } s \text{ to } t. \]
- Example:
  
  \[
  \begin{align*}
  c_a &= 4 \\
  c_b &= 3 \\
  c_c &= 5 \\
  c_d &= 12
  \end{align*}
  \]
  Cost along $b$ and $e$: 8
  Cost without $e$: 3
  Cost of cheapest path without $e$: 15 (along $b$ and $d$)
  Difference is payment: $-15 - (-3) = -12$
  I.e., owner of arc $e$ gets payed 12 for using his arc.
- Note: Alternative path after deletion of $e$ does not necessarily differ from original path at only one position. Could be totally different.
Summary

- New preference model: with money.
- VCG mechanisms generalize Vickrey auctions.
- VCG mechanisms are incentive compatible mechanisms maximizing social welfare.
- With Clarke pivot rule: even no positive transfers and individually rational (if nonnegative valuations).
- Various application areas.