Arrow's Theorem

Lemma (Pareto's Weak Ordering)
Assume $F$ satisfies unanimity $A$ and IIA.
Let $\Pi = (\pi_1, ..., \pi_n)$ and $\Pi' = (\pi'_1, ..., \pi'_n)$ be preference profiles with $\pi_i < \pi'_i$ iff $\pi_i < \pi'_i$ for all $i = 1, ..., n$. Let $\prec$ be $F(\Pi)$ and $\preceq = F(\Pi')$.
Then $\pi_i < \pi'_i$ iff $\pi_i < \pi'_i$.

Remark: Very similar to IIA, but move generator.

Visualization:

\[
\begin{array}{c|c|c|c}
1 & 2 & 3 \\
\hline
\pi_1 & \pi_2 & \pi_3 \\
\hline
\pi'_1 & \pi'_2 & \pi'_3 \\
\end{array}
\]

Proof:
By cases:
1) $a + b$ and $a' + b$ 
2) $a + b$ and $a' + b$
3) $a = b$ and $a' = b$ 
4) $a + e$ and $a' + b$

Case 1: $a + b$ and $a' + b$, ($a < b$ iff $a' < b$) 
implies ($a < b$ iff $a' < b$)

Assume that $a < b$. Construct $\Pi''$ starting from $\Pi$ by moving $\pi_j$ just below $\pi_i$ (if $a + \pi_i$) and $\pi_j$ just above $\pi_i$ (if $b + \pi_i$).

\[
\begin{array}{c}
\Pi \\
\hline
\Pi'' \\
\hline
\Pi'' \\
\hline
\Pi'' \\
\end{array}
\]

Let $\Pi' = F(\Pi'')$. By IIA and our assumption $a < b$, $a' < b$. By transitivity we get $a'' < b$. By IIA, we get $a'' < b$. By symmetry, we can show that $a'' < b$ implies $a < b$. So we get $\Pi''$:

\[
\begin{array}{c}
\Pi \\
\hline
\Pi' \\
\hline
\Pi'' \\
\hline
\Pi'' \\
\end{array}
\]

Case 2: ($a < b$ iff $a' < b$) implies ($a < b$ iff $a' < b$) under the assumption that $a = b$ and $a' = b$.

Case 3: ($a < b$ iff $b < a$) implies ($a < b$ iff $b < a$).

Let $c$ be a third alternative. Construct $\Pi''$ from $\Pi$ by purely $c$ directly below $b$. Construct $\Pi'''$ from $\Pi''$ by moving $c$ directly below $b$. Construct $\Pi'''$ from $\Pi''$, by moving $c$ directly below $b$.

Case 3: homework.

Obviously, ($a < b$) iff $b < a$. By assumption, we have $b < a$ iff $b < a$.

Note: Step 4 implies (by case 1): $a < b$ iff $a < c$

Step 7 implies $b < c$ iff $b < c$

Step 3: $a < c$ iff $b < a$

From this it follows $a < b$ iff $b < a$.

Case 3: homework.
Theorem (Arrow)
Every social welfare function over more than 2 alternatives that satisfies unanimity and IR is dictatorial.
Proof: Consider two alternatives c, d ∈ A, and let us construct a sequence of profiles π^i (i = 1, ..., n) of preference orders such that in π^i exactly the first i voters prefer c over d.

We will show: c^* is a dictator.
Consider two alternatives c, d ∈ A and show that for all j, ..., e ∈ L, \[ c <^j e \] implies c < d, where \[ < = F(\pi^i, ..., \pi^n) \].
Consider e ≠ c, d, and construct a new preference profile \( \pi^i' = (\ldots, <^i, \ldots, <) \):
Depending on whether c <^i d, the table:

\[
\begin{array}{cccc}
j < i^* & e <^j c <^j d & e <^j d <^j c \\
\hline
j = i^* & c <^i e <^i d & \text{Not possible} \\
\hline
j > i^* & c <^i d <^i e & d <^i c <^i e \\
\end{array}
\]

Let \( <^i = F(\pi^i) \). By II A, we have that c < d iff c <^i d.
Let us take a look what preferences once we have between (c, e) and (c, d).

\[
\begin{array}{cccc}
\pi^1 & \pi^i & \pi^1 & \pi^i \\
\hline
\vdots & e <^1 c <^1 a <^1 b & e <^1 c <^1 d <^1 b \\
\hline
\vdots & e <^1 a <^1 c <^1 d & e <^1 d <^1 a <^1 b \\
\vdots & c <^1 f <^1 b <^1 e & e <^1 d <^1 a <^1 b \\
\vdots & c <^1 e <^1 b <^1 d & e <^1 d <^1 a <^1 b \\
\vdots & c <^1 c <^1 d <^1 e & e <^1 d <^1 a <^1 b \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

If we use hand

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By transitivity, we get $c < d$. With \[ I \leq d \]
we get $c < d$. And this means $c < d$ implies $c < d$. Thus $i^*$ is a dictator. \[ \square \]