Grim strategy

Detecting strategy

Tid for dead
Is the given strategy a NE?

\[ V_i(0, g, g) = 3 + 3 \cdot \frac{S}{1 - S} = 3 \cdot (1 + S + S^2 + \ldots) \]

\[ O(g, g) = \langle (c, c), (c, c), \ldots \rangle \]

\[ \sum_{t=1}^{\infty} S^t = \frac{1}{1 - S} \]

\( g' \) - choose D at some point once

\[ \langle (c, c), (c, c), \ldots, (c, D), (D, \ldots), (D, \ldots), (D, \ldots), \ldots \rangle \]

Player 2 has to play D to get the most out of it.

\( g'' \) - choose D at some point and then every other

\( g'' \) against \( g' \) is better than \( g' \) against \( g'' \)
\[ V_2 \left( 0 \left( g, g'' \right) \right) = ? \]

Let's look at \( g'' = c \)

\[ V_2 \left( 0 \left( g, c \right) \right) = 4 + 1 \cdot g + 1 \cdot g^2 + \ldots \]

\[ V_2 \left( 0 \left( g, g \right) \right) = 3 + 3 \cdot g + 3 \cdot g^2 + \ldots \]

\[
4 + \frac{g}{1 - g} \leq 3 + \frac{3g}{1 - g}
\]

\[
1 + \frac{g}{4 - g} \leq \frac{3g}{1 - g}
\]

\[
1 \leq \frac{2g}{1 - g}
\]

\[
1 - g \leq 2g
\]

\[
1 \leq 3g
\]

\[
\frac{1}{3} \leq g
\]

if this is good for then there is no incentive to choose from \( g \) do \( c \)

if \( g \geq \frac{1}{3} \), \( g \) is at least as good as \( c \)
This means \((g,g)\) is a NE!

[If \(S\) is large enough]

What about \((d,d)\)?

\(\rightarrow\) is also a NE.

What about \((t,t)\)

\(\rightarrow\) is also a NE.

Positive message: In repeated games, there are other NEs than the \((D,D)\).

Negative message: Which one do play?
Social Choice Theory

- Aggregation of preferences of group members
- Voting and voting protocols
  - Elections
  - Committee decisions
  - European Council
Def (Social welfare functions and social choice functions)

Let \( A \) be a finite set of alternatives (candidates) and \( L \) be the set of linear orders over \( A \).

For \( n \) voters, \( F : L^n \to L \) denotes a social welfare function and \( F : L^n \to A \) a social choice function.

Notation: A linear order \( \prec \in L \) is called a preference relation. For a voter \( i \), \( \prec_i \) is the preference relation for \( i \). For example, \( a \prec_i b \) means voter \( i \) prefers \( b \) over \( a \).
Example:
Three voters: 1, 2, 3
Candidates: a, b, c

\[
\begin{array}{ccc}
1 & 2 & 3 \\
q & b & c \\
b & c & a \\
c & a & b \\
\end{array}
\]

b \prec_1 a, c \prec_1 a, c \prec_1 b, c \prec_2 b, a \prec_2 b

If we have many voters, a table can specify how many voters agree on one list.

\[
\begin{array}{ccc}
\text{# voters} & 8 & 2 \\
a & b & c \\
b & c & a \\
c & a & b \\
\end{array}
\]

- number of voters
Voting protocols

1) What the voter has to submit (and when).
2) How to compute the result.

- Plurality (aka "first-past-the-post", "winner-takes-it-all"): voters submit only their top preference. The candidate with most votes wins.

  Drawback: winner might not have more than 50% of the vote.

- Plurality with runoff: voters submit top preference. The two candidates with most votes go to a second round.

  In the second round, voters decide between the two candidates.
- Instant runoff voting (rankable votes)
  - each voter gives full preference list
  - iteratively candidates with the least number of top preferences are eliminated until only one candidate remains.

- Borda count
  - each voter submits his preference order
  - if a candidate is in position i of a voter's list, he gets n-i points from that voter.
  - points are added up
  - the one with the most points wins.
A Condorcet winner

- each voter submits his preferences
- perform pairwise comparisons between candidates (how many voters prefer the one over the other)
- if one candidate wins all the comparisons, this is the winner

Drawback: This winner might not exist because the resulting order is cyclic.

In general, all methods might come up with draws, but this is usually ignored because in large elections it is very unlikely.
Running Example:

23 voters, candidates: a, b, c, d, e

<table>
<thead>
<tr>
<th># Votes</th>
<th>8</th>
<th>6</th>
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Plurality: e wins, because e has 8 votes (out of 23)
**Running Example:**

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8 x a < e  
6 x e < a  
4 x e < a  
3 x e < a  
1 x e < a  
1 x a < e  

9 x e is win  
14 x a is win

Plurality with run-off

1st round: e and a
2nd round: 9 x e wins 14 x a wins

⇒ a is the winner
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Instant runoff
1st round: eliminate d
2nd round: eliminate b
3rd round: eliminate a
Now e has 15 versus 8 for c!
→ e wins
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Borda count

a: $8 \times 0 + 6 \times 4 + 4 \times 1 + 3 \times 1 + 1 \times 2 + 1 \times 0 = 33$

b: $= 62$
c: $= 50$
d: $= 46$
e: $= 39$

→ b wins
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In all pairwise comparison b wins

→ b is a Condorcet
Differe d winners for differ e voty protocols!

Which one should be used?

Choice of the voty protocol can be used strategically.

Schulze method

Condorcet method

Has a technique for breaking

University, Private party, others use it