Repeated Games

- If a game is played repeatedly, with
  the same players, players might behave
differently than in one-shot games.

- Fininitely repeated: you play the game
  for a known number of rounds.

- Infinitely repeated: infinite number of rounds.

- Indefinitely repeated: you have given probability
  p that the current game is the last one.
Def.

Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game; let $A = \bigcap_{i \in N} A_i$. A repeated game with $k \in \mathbb{N} \cup \{\infty\}$ moves is an extensive game with perfect information and simultaneous moves $\Gamma = \langle N, A, H, P, (v_i) \rangle$ with

$H = \{\emptyset \} \cup \left( \bigcup_{t=1}^{k} A^t \right) \cup A^k$

$P(h) = N$ for each non-terminal history $h \in H$

If $k \in \mathbb{N}$:

$v_i(h) = \sum_{t=1}^{k} u_i(a^t)$ for $a^1a^2\ldots a^k = h$ for all terminal histories $h$

If $k = \infty$:

Let $S \in J_{0, N}^{\infty}$ be given

$v_i(h) = \sum_{t=1}^{\infty} S^t \cdot u_i(a^t)$ for $a^1a^2\ldots = h$
Example: \( \mathcal{C} = \frac{1}{2} \) \( \psi_i (\phi^i) = 1 \)

\[
\sum_{t=n}^{\infty} \mathcal{S}^t \cdot \psi_i (n) = \sum \left( \frac{\lambda}{2} \right)^t, \quad \lambda = \frac{1}{2} + \frac{\eta}{4} + \frac{\eta}{8} + \ldots \]
Finite Repetition of a Game with Unique Equilibrium

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<th>D (confess)</th>
<th>C (confess)</th>
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<tr>
<td>D</td>
<td>1, 1</td>
<td>4, 0</td>
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<tr>
<td>C</td>
<td>0, 4</td>
<td>3, 3</td>
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- Playing 2 rounds of PD, will we have a different NE?}

- Playing this 100 rounds, DD always is the SPE.

→ For finite repetitions, we always have the same unique equilibrium.
Infinitely repeated games

- Finitely repeated games are rare and you do not interesting results.

- Discounting: What is the $d + d^2 + d^3 + \cdots = ?$
  - it converges to $\frac{d}{1-d}$ for all values $0 < d < 1$.
  
  Proof:
  
  $x = S + dS + d^2S + \cdots$
  
  $x = S + d(x) + d^3(x)$
  
  $x = S + dx$
  
  $x - dx = S$
  
  $(1-d)x = S$
  
  $\frac{x}{1-d} = S$
  
  $x = \frac{S}{1-d}$
Strategies in infinite games:
Finite automata (Moore)

Example

Grim strategy

Tit-for-Tat strategy

Defecting strategy

Action input: profile of actions
Output: strategy
Action to play
\[ V_i(g, g) = \frac{S}{1 - S} \times 3 \] is the overall utility of the repeated game.