Complexity of Solving Strategic Games

The basic problem:

**NASH**: Given a finite 2-player strategic game \( G \),
find a mixed strategy profile \((\sigma, \pi)\) that is a NE of \( G \) if one exists, else return "no".

**SAT**: Given a Boolean formula \( \phi \) in CNF,
find a truth assignment that makes \( \phi \) true
if one exists, else return "no".

In this form, \( \text{NASH} \) looks similar to other search problems, e.g.:

**SAT**:
Given a Boolean formula \( \phi \) in CNF,
find a truth assignment that makes \( \phi \) true
if one exists, else return "no".

**PPAD**: class of search problems that can be polynomially reduced to \( \text{END-OF-LINE} \).

**END-OF-LINE**: Consider a directed graph with
node set \( \{0, 1, 2, \ldots, n\} \) such that each node
has out- and in-degree at most \( 2 \). The graph
is specified by two poly-time functions
\( f \) and \( g \):

\[ f(v) \text{ :\: \text{ successor candidate of } } v \text{ \text{ or empty} } \]
\[ g(v) \text{ :\: \text{ predecessor candidate of } } v \text{ \text{ or empty} } \]

In the graph there is an arc \( v \rightarrow v' \) if and only if \( f(v) = v' \) and \( g(v) = v \).
Given a source node \( s \) in the graph, find some node \( t \) such that \( t \) has in-degree 0 or
in-degree 1, source.

**Polynomial Purity**

**Argument in Directed Graphs**

**Examples**:

Notice:
- \( \text{TFP} \subseteq \text{PPAD} \subseteq \text{TFNP} \subseteq \text{TFD} \)
- Lenstra-Haken algorithm has exponential
  time complexity in the worst case.

**Theorem** (Daskalakis et al., 2006)
\( \text{NASH} \) is \( \text{PPAD} \)-complete.

**2^nd \( \text{NASH} \)**: Given a finite 2-player game \( G \)
and a NE of \( G \), find a second NE of
\( G \) if one exists, else return "no".

**Theorem**
- **2^nd \( \text{NASH} \)** is \( \text{TFP} \)-complete.

Proof Idea:
- Reduction from SAT
Some further results: Given a finite 2-player game $G$, it is difficult to decide whether there exists a NASH $(x, y)$ in $G$ that has one of the following properties:

(a) player 1 (or 2) receives a payoff $\geq k$. -> Guaranteed payoff problem
(b) $U_2(x, y, z) \geq U_2(x, y, z')$ for all $z, z'$. -> Guaranteed social welfare problem
(c) $(x, y)$ is Pareto-optimal, i.e., there is no strategy profile $(x', y')$ such that

$$U_1(x', y) \geq U_1(x, y) \text{ for both } i \in \{1, 2\}, \text{ and }$$
$$U_2(x', y') \geq U_2(x, y') \text{ for at least one } i \in \{1, 2\}.$$

(d) player 1 (or 2) plays some given action with probability $> 0$.

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**Extensive Games**

So far: only simultaneous, one-shot games

**Question:** How to model the sequential structure of many games (e.g., chess?)?

**Approach:** Use extensive games (a = game tree)

**Idea:** Players have several choice points where they can decide how to play. Strategies then map choice points to applicable actions.

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**Definition:** An extensive game with perfect information (EGWI) is a tuple $\Gamma = \langle N, A, H, P, (u): E \rangle$ where:

- $N$ is a finite, nonempty set of players,
- $A$ is a nonempty set of actions,
- $H$ is a set of (finite or infinite) sequences over $A$ (called histories) such that:
  - the empty sequence $<> \in H$;
  - if $<a_1, a_2, \ldots, a_k> \in H$ for some $K \in N \cup \{0\}$ and $L \leq K$, then $<a_1, a_2, \ldots, a_L> \in H$;
  - if $<a_1, a_2, \ldots, a_k>$ is an action sequence such that $<a_1, a_2, \ldots, a_L> \in H$ for each $L \in N$, then $<a_1, a_2, \ldots, a_k> \in H$.

**Assumptions:**

- All the ingredients of $\Gamma$ are common knowledge among the players of the game.
- The set of terminal histories is denoted by $T$.
- $P : H \times Z \rightarrow N$ is the player function assigning to each non-terminal history $h$ in $H$ a player $P(h)$ whose turn it is to move after $h$.
- For each player $i \in N$, $u_i : Z \rightarrow R$ is player $i$'s utility function.

**Some terminologies:**

- $\Gamma$ is finite if $H$ is finite.
- $\Gamma$ has finite horizon if $H$ contains no infinite histories.

**Examples:**

- $\Gamma$ is finite if $H$ is finite.
- $\Gamma$ has finite horizon if $H$ contains no infinite histories.
Example (Sharing Game): Two players have to share two indistinguishable objects.
- Player 1 proposes an allocation.
- Player 2 accepts or declines the proposal.

Game tree:

Formally, \( T = \langle N, A, H, \Pi, (\mu_i)_{i \in N} \rangle \) where

- \( N = \{ 1, 2 \} \)
- \( A = \{ (1,0), (1,1), (2,1), \varnothing, 3 \} \)
- \( H = \{ (>) , \langle (2,0) \rangle , \langle (2,1) \rangle , \langle (0,2) \rangle , \langle (2,0), 3 \rangle , \langle (2,1), \varnothing \rangle , \langle (1,4), \varnothing \rangle , \ldots \} \)
- \( Z = \{ h \in H : |h| = 2 \} \)
- \( \Pi(>) = 1, \quad \Pi(\varnothing) = 2 \quad \forall h \in H \setminus \{ (2,0), 3 \} \)
- \( \mu_1((2,0), \varnothing) = 2, \quad \mu_1((1,1), 1) = 0 \)