Existence of MSNE

Theorem (Nash): Every finite strategic game has a MSNE.

Proof: later

Preliminaries.

Def.: (a) $X \subseteq \mathbb{R}^n$ is bounded if for

$M \leq i \leq m$ exist $a_i, b_i \in \mathbb{R}$ such that

$X \subseteq \prod_{i=m}^m [a_i, b_i]$.
Counterexample:

(b) \( X \subseteq \mathbb{R}^n \) is **closed** if the limit of each convergent sequence of elements of \( X \) is contained in \( X \).

Counterex: \([0,1)\) not closed

Sequence: \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rightarrow 1 \notin [0,1) \)
(c) $X \subseteq \mathbb{R}^n$ is convex if for each $x, y \in X$, and for any $t \in [0,1]$, then $(1-t) \cdot x + t \cdot y \in X$.

Counterex.: 

(d) For function $f : X \rightarrow 2^X$, the graph $\text{Graph}(f)$ is the set $\text{Graph}(f) = \{ (x, y) \mid x \in X, y \in f(x) \}$. 
Theorem (Kakutani): Let \( X \subseteq \mathbb{R}^n \) be nonempty, closed, bounded, and convex set and \( f: X \to 2^X \) be a function such that:

(i) for each \( x \in X \), the set \( f(x) \subseteq X \) is nonempty and convex, and

(ii) Graph \( (f) \) is closed.

Then, \( \exists x \in X \) with \( x \in f(x) \), i.e. \( f \) has a fixed point. \( \square \)
Example (c): (a) \( f : [0, 1] \rightarrow 2^{[0, 1]} \), \( f(x) = \{ y \mid y \leq x \} \).

\[ \text{Graph}(f) \]
\[ x \in f(x) \]
\[ (e.g. \quad \frac{1}{2} \in \{ y \mid y \leq \frac{1}{2} \} ) \]

(b) \( f : [0, 1] \rightarrow 2^{[0, 1]} \), \( f(x) = \{ y \mid 1 - \frac{x}{2} \leq y \leq 1 - \frac{x}{4} \} \).

\[ \text{Graph}(f) \]
\[ f(0) = [1, 1] \]
\[ f(1) = [1 - \frac{1}{2}, 1 - \frac{1}{4}] \]