Existence of MSNE

Theorem (Nash): Every finite strategy game has a MSNE.

Proof: Later

Preliminaries

Def: (a) $X \subseteq \mathbb{R}^n$ is bounded if for all $i \leq n$, $a_i, b_i \in \mathbb{R}$ and that $X \subseteq \bigcap_{i=1}^{n} [a_i, b_i]$.

(b) $X \subseteq \mathbb{R}^n$ is closed if the limit of each convergent sequence of elements of $X$ is contained in $X$.

Counterexample: $\frac{1}{i}$: \[\begin{array}{c}
\text{Seq: } 1, 1/2, 1/3, 1/4, \ldots \xrightarrow{\text{conv}} 0 \\
\text{Lim } 0 \not\in [0, 1)
\end{array}\]

(c) $X \subseteq \mathbb{R}^n$ is convex if for each $x, y \in X$, and for any $t \in [0, 1]$, also $(1-t) x + ty \in X$.

Counterexample: $\text{Graph}\{x, y\}$

(d) For function $f: X \to 2^X$, the set $f$ is the set $\text{Graph}(f) = \{(x, y) \mid x \in X, y \in f(x)\}$.

Theorem (Kakutani): Let $X \subseteq \mathbb{R}^n$ be nonempty, closed, bounded, and convex set and $f: X \to 2^X$ a function such that:

(i) For each $x \in X$, the set $f(x) \subseteq X$ is nonempty and convex, and

(ii) Graph $(f)$ is closed.

Then, there exists $x \in X$ with $x \in f(x)$, i.e., $f$ has a fixpoint.
Example: (a) \( f : [0, 1] \to 2^{[0, 1]} \), \( f(x) = \{ y \mid y \leq x \} \).

Graph \( f \)

\( x \in f(x) \)
(e.g., \( \frac{1}{2} \in \{ y \mid y \leq \frac{1}{2} \} \))

(b) \( g : [0, 1] \to 2^{[0, 1]} \), \( g(x) = \{ y \mid 1 - \frac{x}{2} \leq y \leq 1 - \frac{x}{4} \} \).

Graph \( g \)

\( g(x) = [1, 1] \)
\( g(x) = [1 - \frac{1}{2}, 1 - \frac{1}{4}] \)