(b) **Li-N (I):** \( U_n(x^*, y^*) = \max_{x \in A_n} \min_{y \in A_l} U_n(x, y) \).

\[ U_n(x^*, y^*) = -U_l(x^*, y^*) = -\max_{y \in A_l} \min_{x \in A_n} U_l(x, y) \]

\[ = \min_{y \in A_l} \max_{x \in A_n} U_n(x, y) \]

(c) Let \( x^*, y^* \) be NE of games 1 and 2, and \( \max_{x \in A_n} \min_{y \in A_l} U_n(x, y) = \min_{y \in A_l} \max_{x \in A_n} U_n(x, y) \)

\[ \Rightarrow -u^* = \max_{y \in A_l} \min_{x \in A_n} U_n(x, y) \]

\[ \Rightarrow u^* = \max_{y \in A_l} \min_{x \in A_n} U_l(x, y) \]

\[ \Rightarrow x^*, y^* \text{ is a NE} \]

\[ \text{Mixed strategies} \]

**Motivating example:** Mating preferences

<table>
<thead>
<tr>
<th>( \frac{1}{3} )</th>
<th>( \frac{2}{3} )</th>
<th>( 1, -1 )</th>
<th>(-1, 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2}{3} )</td>
<td>( H )</td>
<td>( -1, 1 )</td>
<td>( 1, -1 )</td>
</tr>
</tbody>
</table>

For player 1: \( \alpha_n(H) = \frac{2}{3} \), \( \alpha_n(T) = \frac{1}{3} \)
For player 2: \( \alpha_l(H) = \frac{1}{3} \), \( \alpha_l(T) = \frac{2}{3} \)
Define: Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game. Let $\Delta(A_i)$ be the set of probability distributions on $A_i$.

Then, an $\alpha_i \in \Delta(A_i)$ is a **mixed strategy** of player $i$ in $G$, where $\alpha_i(a_i)$ is the probability that player $i$ chooses $a_i \in A_i$.

A profile $(\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$ induces a probability distribution on $A = \prod_{i \in N} A_i$ as $\rho(a) = \prod_{i \in N} \alpha_i(a_i)$.

\[\rho(H, H) = \frac{2}{5} \cdot \frac{4}{5} = \frac{8}{25}, \quad u_1(H, H) = 1\]
\[\rho(H, T) = \frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25}, \quad u_1(H, T) = -1\]
\[\rho(T, H) = \frac{3}{5} \cdot \frac{4}{5} = \frac{12}{25}, \quad u_2(T, H) = -1\]
\[\rho(T, T) = \frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25}, \quad u_2(T, T) = 1\]

$u_1$, expected: \( \frac{2}{5} \cdot (1) + \frac{4}{5} \cdot (-1) + \frac{1}{5} \cdot (-1) + \frac{2}{5} \cdot (+1) = \frac{4}{5} \)

$u_2$, expected: \( \frac{4}{5} \)

**Definition:** Let $\alpha_i \in \prod_{i \in N} \Delta(A_i)$. Then the **expected utility** of $\alpha_i$ for player $i$ is

\[U_i(\alpha_i) = \sum_{a \in A} \rho(a) \cdot u_i(a) \]

Example: $U_1(\alpha_1, \alpha_2) = -\frac{4}{5}$
$U_2(\alpha_1, \alpha_2) = +\frac{4}{5}$

**Definition:** The **mixed extension** of a (finite) strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is the strategic game $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$ where the $\Delta(A_i)$ and $U_i$ are defined as above.

**Definition:** Let $\alpha_i$ be a mixed strategy. The **support** of $\alpha_i$ is the set of pure strategies

\[\text{supp}(\alpha_i) = \{a_i \in A_i \mid \alpha_i(a_i) > 0\}\]
Def: Let \( G \) be a strategic game. A **Nash-equilibrium** in mixed strategies (mixed-strategy NE, MSNE) of \( G \) is a NE of the mixed extension of \( G \).

Support-Lemma: Let \( G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \) be a finite strategic game. Then \( \alpha^* = (\alpha^*_1, \ldots, \alpha^*_n) \in \prod_i A_i \) is a MSNE of \( G \) if

for each player \( i \in N \), every pure strategy in \( \text{supp}(\alpha^*_i) \) is a best response to \( \alpha^*_-i \).

Proof (Support Lmm.): Let \( \alpha^* \) be a MSNE and \( \alpha^*_i \in \text{supp}(\alpha^*_i) \).

\( \Rightarrow ^e \): Suppose that \( \alpha^*_i \) is not a best response to \( \alpha^*_-i \). Then \( \alpha^*_i, \alpha'_i \in \Theta_i \) s.t.

\[ U_i(\alpha^*_i, \alpha'_i) > U_i(\alpha^*_i, \alpha^*_i). \]

Then \( \alpha'_i \) with weight shifted from \( \alpha^*_i \) to \( \alpha'_i \) would be a better response to \( \alpha^*_-i \) than \( \alpha^*_i \) is. Then \( \alpha^*_i \in \mathcal{B}_i(\alpha^*_i). \) Then \( \alpha^*_i \) is not a MSNE. \( \Box \)

Example: Mutually Penetrating (\( p = 0.6 \))

\[
\begin{align*}
\alpha_0(H) &= \frac{2}{3}, & \alpha_0(T) &= \frac{1}{3} \\
\alpha_1(H) &= \frac{4}{3}, & \alpha_1(T) &= \frac{2}{3}
\end{align*}
\]

MSNE?

\[
\begin{align*}
U_2(\alpha_0, H) &= \alpha_0(H) \cdot u_1(H, H) \\
&\quad + \alpha_0(T) \cdot u_1(T, H)
&= \frac{2}{3} \cdot (1) + \frac{1}{3} \cdot (2) = -\frac{1}{3}
\end{align*}
\]

\[
\begin{align*}
U_2(\alpha_1, T) &= \alpha_0(H) \cdot u_2(H, T) \\
&\quad + \alpha_0(T) \cdot u_2(T, T)
&= \frac{4}{3} \cdot (1) + \frac{2}{3} \cdot (1) = \frac{2}{3}
\end{align*}
\]

\( \Rightarrow (\alpha_0, \alpha_1) \text{ not a MSNE.} \)